

# ON THE ERGODIC THEORY OF CERTAIN NON-DISCRETE ACTIONS AND TOPOLOGICAL ORBIT EQUIVALENCES

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*To Professor Yu.S. Il'yashenko, on the occasion of his 70<sup>th</sup> birthday*

**ABSTRACT.** Quasi-invariant measures for non-discrete groups of diffeomorphisms containing a Morse-Smale dynamics are studied. The assumption concerning the presence of a Morse-Smale dynamics allows us to extend to higher dimensions a number of recently established results for non-discrete groups acting on the circle. These results are also applied to show that, for many groups as above, every continuous orbit equivalence must coincide almost everywhere with a diffeomorphism of the corresponding manifold.

## 1. INTRODUCTION

Essentially this paper consists of results about the ergodic theory of certain *non-discrete* group actions along with a number of related questions, some of them sort of loosely stated. Motivations for this study stem from a few sources and this Introduction is devoted to describing some of them by following a chronological order. Statements for the main result of this paper are deferred to Section 2. Yet, to illustrate the problems we shall be dealing with, let us state an easy “linear” version of these results.

**Linear Theorem (corollary of Corollary B and of Theorem D in Section 2).** *Consider the standard projective action of  $\mathrm{PSL}(2, \mathbb{C})$  on the sphere  $S^2$ . Suppose that  $G_1, G_2$  are countable dense subgroups of  $\mathrm{PSL}(2, \mathbb{C})$ . Then we have:*

- (1) *Suppose that  $\mu$  is a  $d$ -quasiconformal probability measure for  $G_1$ . Then  $\mu$  is absolutely continuous and, in particular,  $d = 2$ .*
- (2) *Suppose that  $h$  is a topological orbit equivalence between the actions of  $G_1$  and  $G_2$ . Then  $h$  is equivariant and it coincides with an element of  $\mathrm{PSL}(2, \mathbb{C})$ .*

The conformal character of the projective action of  $\mathrm{PSL}(2, \mathbb{C})$  on  $S^2$  is not particularly important for the above theorem. In fact, similar statements hold not only to isometry groups of higher dimensional hyperbolic spaces but also to the projective action of  $\mathrm{PSL}(n, \mathbb{C})$  whose ergodic theory is less developed.

To abridge the discussion in this Introduction, let  $\mathrm{Diff}(M)$  denote the group of diffeomorphisms of a compact manifold  $M$  without specifying the regularity: the reader may consider these diffeomorphisms to be real-analytic though this is not always necessary. The dynamical study of actions associated to non-discrete representations of (finitely generated) groups has become more of a focus of interest in the past two decades. In the so-called “linear case”, i.e. when the action is simply the restriction of a Lie group action to a countable subgroup  $G$ , it was soon realized that the “closure”  $\overline{G}$  of  $G$  (inside the mentioned Lie group) contains a wealth of information on the dynamics of  $G$  itself. Whereas the study of “non-linear dynamics” is much harder, in certain cases it is possible to define a dynamically meaningful closure for a countable non-discrete group  $G$  acting on some space. To the best of my knowledge, the first time that

these ideas were effectively put to work was in the papers by Shcherbakov [Sh-1], [Sh-2] naturally inspired by the seminal work of Y. Il'yashenko [Il-1] on dynamics of complex polynomial vector fields. In his papers, Shcherbakov deals with the dynamics of non-solvable subgroups of  $\text{Diff}(\mathbb{C}, 0)$ . Independently and slightly later than Shcherbakov, the same idea was used by Nakai [Na] who elegantly established accurate versions of Shcherbakov results. This collection of results, that deserves to be called Shcherbakov-Nakai theory, clarifies the topological dynamics and the basic ergodic theory of the corresponding group actions (or rather pseudogroup actions). This theory however falls short of settling some questions belonging to the “higher ergodic theory” of these actions, as it will be seen below. Subsequently to their work, Ghys was the first to explicitly talk about “non-linear and non-discrete” group actions [G]. In particular, it is clear from his work that “non-solvable subgroups of  $\text{Diff}(\mathbb{C}, 0)$ ” are necessarily non-discrete and this provided a kind of explanation for the existence of Shcherbakov-Nakai theory. It is also shown in [G] that non-discrete finitely generated subgroups of  $\text{Diff}(M)$  are quite common and the general idea of defining dynamical closures for the corresponding actions is implicitly advanced. Basically, it amounts to constructing non-trivial vector fields on  $M$  whose (local) flows can be approximated by actual elements in  $G$ . When these vector fields exist, they are said to belong to the closure of  $G$  (cf. Section 2 for an accurate definition).

For a manifold not reduced to a single point, the program proposed by Ghys was satisfactory carried out on the circle by the author in [R-1]. Whereas no general result is available for higher dimensional manifolds, the method of [R-1] still works in the presence of some “local contraction”, for example on a neighborhood of a point  $p \in M$  that happens to be a contracting hyperbolic fixed point for some element  $f \in G$ . This type of idea was further developed in [L-R] (see also [Be-1]) and used to extend Il'yashenko's theory [Il-1] of (singular) holomorphic foliations on projective spaces. Around the same time, and again independently, Belliard [Be-2] and the author [R-3] noted that the theory of [R-1] generalizes to non-discrete groups  $G \subset \text{Diff}(M)$  containing a Morse-Smale diffeomorphism (satisfying some very generic conditions). Though Morse-Smale dynamics does exist in a number of interesting cases, the corresponding results appear to be much less known than their analogues for  $\text{Diff}(\mathbb{C}, 0)$  and  $\text{Diff}(S^1)$ . Hopefully the discussion in this paper will help to raise further interest in the potential applications of these techniques.

Having briefly mentioned the general method of exploiting the “closure of a non-discrete subgroup  $G$  of  $\text{Diff}(M)$ ” to study the dynamics of  $G$  itself, let us now turn our attention towards problems belonging to the “higher ergodic theory” of  $G$ . Here, it may be noted that the usefulness of the above mentioned vector fields lies primarily in the fact that they allow us to prove *in the non-linear setting*, results that are easy in the linear one. For example, these vector fields are effective tools to show denseness of orbits and ergodicity with respect to the Lebesgue measure for the corresponding groups (the corresponding results being very well-known in the “linear” setting). However, there are subtle questions about dynamics that are hard even in the “linear” case, and these include the description of quasi-invariant measures and the rigidity/flexibility of the orbit structure (in the sense of orbit equivalence, cf. below). These questions are the prototype of what we have vaguely called the “higher ergodic theory” of group actions. The difficulty of describing quasi-invariant measures was discussed in [R-5] and it is illustrated by the following theorem due to Kaimanovich and Le Prince [K-LP]: for every Zariski-dense finitely generated subgroup  $G$  of  $\text{PSL}(2, \mathbb{R})$  there is a non-degenerate probability measure  $\nu$  on  $G$  giving rise to a singular stationary measure  $\mu$  for the corresponding (projective) action on  $S^1$ . Yet, considering the closure of  $G$  led us to new insight in the structure of quasi-invariant measures for non-discrete (and non-solvable) subgroups of  $\text{Diff}(S^1)$ . Another motivation for this paper was then to show that most of the results in [R-5] can be generalized to higher dimension in the presence of Morse-Smale dynamics.

Finally a last motivation for the discussion conducted in this paper has to do with “trajectory theory”, in the spirit of [C-S-V], for the corresponding group actions. Among the group actions constructed in this paper (corresponding essentially to free group actions), we consider whether or not they are topologically orbit equivalent (topologically OE) with respect to the quasi-invariant Lebesgue measure. It is then shown that the existence of a topological OE implies that the actions are, indeed, conjugate in  $\text{Diff}^\omega(M)$ . It may also be pointed out that continuous OE appears to be related to the uniqueness of certain cross products over  $C^*$ -algebras, see [G-P-S]. Note that, in the measurable category, OE bears a close connection with von Neumann algebras and this connection has inspired a vast literature (see the survey [Fu]). In the direction of rigidity results, we shall also sketch a proof of the fact that an *equivariant OE* (i.e. a measurable conjugacy) between groups as before must coincide a.e. with a diffeomorphism of  $M$ . A few questions about general OE will also be raised.

Let us close this Introduction with a brief outline of the structure of this work. The first thing to be noted is that this structure is not fully linear: some points are developed to a level of generality greater than what is needed for the use made in this paper. These situations happen when we want to single out additional questions or when we believe the corresponding results are interesting in themselves (in the latter case, see the material in Section 3). This said, Section 2 contains accurate statements for our main results along with the relevant definitions. Sections 3 and 4 deserve further comments. As mentioned, vector fields in the “closure of groups” constitute the main tool used in our discussion. Whereas we are mainly concerned with analytic diffeomorphisms, these techniques are also relevant in the differentiable case, though their extensions beyond the analytic context hardly been considered (apart from [R-4]). Therefore some was made to clarify the existence of this type of vector fields in the  $C^\infty$ -case as well. In fact, dealing with groups of  $C^\infty$ -diffeomorphisms presents us with brand new situations where the notion of vector fields in the “closure of groups” can be brought to shed new light in the dynamics. A remarkable example of it, being the conclusion that Ghys-Sergiescu [G-S] realization of Thompson group in  $\text{Diff}(S^1)$  is “very discrete” in the sense that it does not satisfy condition (1) in Section 2. In view of it, Section 3 contains various results ensuring the existence of vector fields in the “closure” of diffeomorphisms groups (both in the smooth and analytic cases). Proofs are usually harder in the smooth case and are given in full detail. Section 4 begins with some extensions of the results in Section 3 for the analytic case. We shall not come back to the smooth case since we consider that, after the discussion in Section 3, the analogous statements for  $C^\infty$ -diffeomorphisms can safely be left to the reader. The section then continues with a crucial result about quasi-invariant measures for groups contained the translations of  $\mathbb{R}^n$ . Then, after a brief discussion on “general” and “generic” (pseudo) Lie algebras associated to groups, our main results about quasi-invariant measures are derived. Finally Section 5 builds on the previous material to discuss some rigidity phenomena and a number of related questions.

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## 2. MAIN RESULTS

Our central statements will involve groups of real analytic diffeomorphisms since essentially all natural examples discussed in this paper belong to this category. However in Section 3 it will become clear that most of these statements holds in the  $C^\infty$ -category as well. Let then  $M$  be a compact real analytic manifold and denote by  $\text{Diff}^\omega(M)$  the group of its real analytic diffeomorphisms. It may be convenient to equip  $\text{Diff}^\omega(M)$  with the analytic topology of Takens so that  $\text{Diff}^\omega(M)$  inherits the Baire property. To define this topology, we first need to fix a *complexification* of  $M$ . For this, choose a realization of  $M$  as an (real analytic) sub-manifold of an Euclidean space  $\mathbb{R}^s$ . Setting  $\mathbb{R}^s \subset \mathbb{C}^s$ , a *complexification* of  $M$  is a complex sub-manifold  $\widetilde{M} \subset \mathbb{C}^n$  containing  $M$  (locally) as  $\mathbb{R}^s$  is contained in  $\mathbb{C}^s$ , see [B-T] for details. We then denote by  $\widetilde{M}_\tau$  the set of points in  $\widetilde{M}$  whose distance to  $M$  is less than  $\tau > 0$ . The analytic topology is defined by stating that a sequence of real analytic maps  $f_i : M \rightarrow \mathbb{R}$  converges to a (necessarily analytic) map  $f_\infty$  if, to every  $\tau, \varepsilon > 0$ , there corresponds  $i_0$  such that  $f_i - f$  has a holomorphic extension  $\widetilde{f_i - f}$  to  $\widetilde{M}_\tau$  satisfying  $\sup_{\widetilde{M}_\tau} \|\widetilde{f_i - f}\| \leq \varepsilon$  whenever  $i > i_0$ . By virtue of Cauchy formula, it follows that convergence in this analytic topology implies  $C^\infty$ -convergence.

Next, let us explain Ghys's criterion for non-discrete groups [G]. Given a (finitely generated) group  $G \subset \text{Diff}^\omega(M)$ , Ghys associates to every (finite) generating set  $S$  of  $G$  a sequence of sets  $S(j) \subseteq G$  defined as follows:  $S(0) = S$  and  $S(j+1)$  is the set whose elements are the commutators written under the form  $[F_1^{\pm 1}, F_2^{\pm 1}]$  where  $F_1 \in S(j)$  and  $F_2 \in S(j) \cup S(j-1)$  ( $h \in S(0)$  if  $j = 0$ ). Then the group  $G$  is said to be *pseudo-solvable* if for some generating set  $S$  as above, the sequence  $S(j)$  becomes reduced to the identity for  $j$  large enough. Since the definition of pseudo-solvable group may depend on the generating set, its geometric content is not clear. However, when  $G \subset \text{Diff}^\omega(M)$ , then it is known in some cases that  $G$  is pseudo-solvable if and only if it is solvable. Furthermore, whenever  $S$  is constituted by diffeomorphisms sufficiently close to the identity in the analytic topology, it is shown that every element in  $S(j)$  admits a holomorphic extension to a uniform neighborhood  $\widetilde{M}_\tau$  of  $M$  in  $\widetilde{M}$ . Besides, the sequence formed by these holomorphic extensions converge uniformly to the identity on  $\widetilde{M}_\tau$ . In particular, these elements form a sequence converging to the identity in the  $C^\infty$ -topology (though they do not converge to the identity in the above defined analytic topology on  $\text{Diff}^\omega(M)$ ).

Going back to pseudo-solvable groups, they are clearly very specific among finitely generated subgroups of  $\text{Diff}^\omega(M)$ . Actually, given  $k \in \mathbb{N}$ ,  $k \geq 2$ , consider the product  $(\text{Diff}^\omega(M))^k$  of  $k$ -copies of  $\text{Diff}^\omega(M)$  endowed with the product analytic topology. Then, by using the perturbation technique introduced in [B-T], the following can easily be proved (cf. [R-3]): there is  $G_\delta$ -dense set  $\mathcal{U}_k \subset (\text{Diff}^\omega(M))^k$  such that the diffeomorphisms  $f_1, \dots, f_k$  generate a free group on  $k$  letters provided that the  $k$ -tuple  $(f_1, \dots, f_k)$  belongs to  $\mathcal{U}_k$ . Combined to Ghys's criterion, this already shows that “non-discrete” subgroups of  $\text{Diff}^\omega(M)$  are quite common: their study is hence justified. However, due to the existence of more subtle examples/applications, for the purposes of this paper, a finitely generated group  $G \subset \text{Diff}^\omega(M)$  will be called *non-discrete* if it contains a sequence of elements  $h_i$ ,  $h_i \neq \text{id}$  for every  $i \in \mathbb{N}$ , converging to the identity in, say, the  $C^\infty$ -topology. Naturally this assumption is weaker than the existence of holomorphic extensions converging to the identity on a uniform domain  $\widetilde{M}_\tau$ , but it enables us to obtain results valid also in the smooth category.

Consider a point  $q \in M$  and an open neighborhood  $U \subset M$  of  $q$ . Denote by  $G_U$  the pseudogroup of diffeomorphisms from open subsets of  $U$  to  $U$  induced by the restrictions of in  $G$ . Then  $G$  is said to be conjugate to a finite dimensional Lie group about  $q$ , if there is a (germ of) finite dimensional Lie group generating a pseudogroup  $\Gamma$  of local transformations on (open

subset of)  $U$  containing a sub-pseudogroup conjugate to  $G_U$  by some (local)  $C^\infty$ -diffeomorphism  $\varphi : U \rightarrow M$  fixing  $q$ .

A fixed point for  $F \in \text{Diff}^\omega(M)$  is said to be contracting (resp. expanding) if all the eigenvalues of  $D_p F$  have absolute value less than 1 (resp. greater than 1). In the rest of this section, we shall consider finitely generated subgroups of  $\text{Diff}^\omega(M)$  satisfying the following conditions:

- (1)  $G$  is a finitely generated ( $C^\infty$ ) non-discrete subgroup of  $\text{Diff}^\omega(M)$ .
- (2)  $G$  contains a Morse-Smale diffeomorphism  $F : M \rightarrow M$ . Furthermore every fixed point  $p$  of some power of  $F$  that is either contracting or expanding must be also non-resonant.
- (3)  $G$  leaves no proper analytic subset of  $M$  invariant.
- (4) If  $U \subset M$  is an open set invariant by  $G$ , then the action of  $G$  on  $U$  does not leave invariant any regular  $C^\infty$ -foliation.
- (5)  $G$  is not conjugate to a finite dimensional Lie group about any of the contracting (or expanding) fixed point of  $F$ .
- (6) Every element  $g \neq \text{id}$  in  $G$  has a set of fixed points of codimension at least 2.

Concerning the above made assumptions, it is clear that condition (1) is totally natural since most of our statements failed to be verified by discrete groups (cf. below). Condition (3) is very minor and intended only to simplify the discussion (note that in dimension 1 the action is often asked to be minimal) and, similarly, Condition (4) is also natural (analogous to low rank phenomena) as it serves to rule out a splitting of the dynamics of  $G$ . The assumption made in condition (5) is again harmless since the corresponding statements can easily be worked out when  $G$  is locally conjugate to a finite dimensional Lie group about a contracting/expanding fixed point of  $F$ . The assumption made in condition (6) plays a role only in the proof of Theorem D which uses the results of [F-W]. However, it is not known whether or not Condition (2) is intrinsic to the problems or just a superfluous assumption needed here for technical reasons (the condition about non-resonance in fixed points is nonetheless a minor one, cf. Remark 3.6). Yet, extending to higher dimensions the methods of [R-1], for example in the area-preserving case, remains a major open problem.

Recall that a measure  $\mu$  on  $M$  is said to be quasi-invariant under  $G \subset \text{Diff}^\omega(M)$  if and only if for every element  $g \in G$  the Radon-Nikodym derivative  $d(g^*\mu)/d\mu$  exists. Naturally this happens if and only if for every  $g \in G$  the measures  $g^*\mu$  and  $\mu$  have the same null-measure sets. In turn, an accurate definition of the closure  $\overline{G}$  of  $G \subset \text{Diff}^\omega(M)$  can be found in Section 3. With the preceding assumptions, the first main result of [R-5] admits the following generalization to higher dimensions.

**Theorem A.** *Suppose that  $G$  satisfies conditions (1) - (5) above. Let  $\mu$  be a probability measure on  $M$  that is quasi-invariant by the closure  $\overline{G}$  of  $G$ . Then  $\mu$  is absolutely continuous.*

Theorem A has a couple of interesting corollaries whose statements require the following definition.

**Definition 2.1.** Let  $\mu$  be a probability measure on  $M$  and consider a group  $G \subset \text{Diff}^\omega(M)$ . Given  $d \in \mathbb{R}_+^*$ , the measure  $\mu$  will be called a  $d$ -quasi-volume for  $G$  if there is a constant  $C$  such that for every point  $x \in S^1$  and every element  $g \in G$ , the Radon-Nikodym derivative  $d\mu/dg_*\mu$  satisfies the estimate

$$\frac{1}{C} \|\text{Jac}[Dg](x)\|^d \leq \frac{d\mu}{dg_*\mu}(x) \leq C \|\text{Jac}[Dg](x)\|^d,$$

where  $\text{Jac}[Dg](x)$  stands for the Jacobian determinant of  $Dg$  at the point  $x$ .

In particular, when the action of  $G$  on  $M$  preserves a conformal structure, the above definition reduces to the usual notion of  $d$ -quasiconformal measures familiar from Patterson-Sullivan theory, see [Su]. The first corollary of Theorem A (paralleling Corollary B of [R-5]) states that groups as in Theorem A do not possess  $d$ -quasi-volumes other than those that are absolutely continuous. Note that, compared to previous statements, the two corollaries below are among the very first results valid for this natural type of quasi-invariant measures beyond the conformal context.

**Corollary B.** *Suppose that  $G$  is as in Theorem A and let  $\mu$  be a  $d$ -quasi-volume for  $G$ . Then  $\mu$  is absolutely continuous and  $d$  equals the dimension of  $M$ .*

The previous results can be applied, in particular, to the foliations constructed in [L-R]. Thus we obtain:

**Corollary C.** *Every (transverse)  $d$ -quasi-volume measure for the foliations constructed in [L-R] is absolutely continuous.*

Considering the ergodic theory of Kleinian groups and, in particular, the prominent role played by Patterson-Sullivan measures, Corollary B seems to indicate that the ergodic theory of non-discrete groups may share properties with (geometrically finite) convex co-compact Kleinian groups. Similarly, in analogy with rational maps, the Lyubich-Minsky laminations associated to the latter may be considered, cf. [L-M]. These laminations carry several types of natural measures, including Patterson-Sullivan type measures, cf. [K-L]. It is then natural to ask if (or when) the holonomy pseudogroup of the laminations in question is discrete. In fact, the pseudogroup of these laminations seems to provide a suitable framework to ask whether or not the “pseudogroup arising from the several inverse branches of a rational map is discrete”. The understanding of discrete/non-discrete situations, along with the ergodic consequences immediately derived in the latter case, would constitute a fine addition to Sullivan’s dictionary, besides shedding additional light in the structure of the measures constructed in [K-L].

Going back to Theorem A, let us also mention that is probably not hard to establish a comparison theorem between quasi-invariant measures with Hausdorff dimension  $d$  and the corresponding  $d$ -dimensional Hausdorff measures analogous to the statement provided in [R-5]. The sequence of this article, however, goes in the direction of rigidity for orbit equivalences where there is a few new questions.

Concerning orbit equivalences and rigidity results, let  $G_1, G_2 \subset \text{Diff}^\omega(M)$  be two finitely generated groups. Suppose also that  $G_1$  is contained in a finite dimensional Lie group effectively acting on  $M$  (or more generally contained in a locally compact subset of  $\text{Diff}^\omega(M)$ ). In other words, the action of  $G_1$  is “linear” according to our previous terminology. Recall that the groups  $G_1, G_2$  are said to be *continuously orbit equivalent* if there is a homeomorphism  $h$  of  $M$  taking orbits of  $G_1$  to orbits of  $G_2$ . Note, in particular, that this definition does not require  $h$  to be equivariant and, in particular, it does not imply that  $G_1, G_2$  are isomorphic. With these definitions, one of our main results can be stated as follows.

**Theorem D.** *Suppose that  $G_1, G_2 \subset \text{Diff}^\omega(M)$ , where the action of  $G_1$  is supposed to be “linear”. Suppose also that  $G_2$  fulfils all the conditions (1) - (4) as well as condition (6) in Section 2. Then every homeomorphism  $h : M \rightarrow M$  realizing an orbit equivalence between the actions of  $G_1, G_2$  on  $M$  is equivariant and coincides with a real analytic diffeomorphism of  $M$ .*

Theorem D will be proved in Section 5 which includes a comparison between Theorem D and some previous related results. Being rather flexible, our methods yield further results whose proofs will only be sketched there to keep the length of the article under control. In particular,

it allows us to conclude that every (measurable) *equivariant* orbit equivalence between  $G_1, G_2$  as above coincides a.e. with an element of  $\text{Diff}^\omega(M)$ . The question on whether or not there exist non-equivariant orbit equivalences between these actions is very interesting and some implications will be mentioned at the end of Section 5. Another question that will only be mentioned in Section 5 has to do with the “linear” assumption imposed on  $G_1$ . If this condition is dropped, the result hinges on deciding whether or not a curious phenomenon of convergence involving the group  $G_1$  can occur.

### 3. CLOSURE OF GROUPS AND VECTOR FIELDS

Albeit some of the material in this section is well-known [L-R], [Be-1], [Be-2] and [R-3], our discussion is mostly original. In fact, a common trace in the mentioned papers is that they deal with “analytic” limits contained in the closure of  $G$  while we consider smooth limits making sense for  $C^\infty$ -groups (a brief discussion of smooth limits appeared previously in [R-4]). Actually the discussion conducted below involves various topologies and their corresponding notions of limits so as to make clear what is valid beyond the “analytic” condition. As a consequence, this section is more comprehensive than what is needed later. Nonetheless I believe that clarifying these issues about topologies makes this discussion worth of interest.

In the sequel  $M$  is a manifold and  $G$  is a finitely generated group of diffeomorphisms of  $M$ . In terms of regularity, it suffices to assume these diffeomorphisms to be of class  $C^\infty$  or real analytic: the discussion is more concerned with the notions of convergence than with the regularity of the diffeomorphisms themselves. Whereas the case  $C^r$  is also sufficient at certain points, the corresponding adaptations are going to be left to the reader.

To begin with, let  $U \subseteq M$  be a connected open set and consider the restriction to  $U$  of elements in  $G$ . Consider also a vector field  $X$  defined on  $U$  and denote by  $\phi^t$  its local flow. The vector field  $X$  is said to be *in the  $C^\infty$ -closure of  $G$  (relative to  $U$ )* if the following holds:

- (a):  $X$  is a  $C^\infty$ -vector field.
- (b): For every  $t \in \mathbb{R}$  and  $V \subset U$  such that  $\phi^s$  is defined on  $V$  for all  $s \in [0, t]$ , there exists a sequence of elements  $g_i \in G$  whose restrictions to  $V$  converge in the  $C^\infty$ -topology to the diffeomorphism  $\phi^t : V \rightarrow \phi^t(V)$ .

Analogously, the vector field is said to be in the  $C^k$ -closure of  $G$  if the sequence  $g_i$  approximates  $\phi^t$  only in the  $C^k$ -topology.

The analytic variant of the closure of  $G$  is however slightly more subtle. Fix a *real analytic* vector field  $X$  defined on  $U$  as above. Then consider a neighborhood  $\tilde{U}$  of  $U$  in  $\tilde{M}$  where  $X$  admits a holomorphic extension denoted by  $\tilde{X}$  whose (real) local flow will still be denoted by  $\phi^t$ . Then  $X$  is said to belong to the  $C^\omega$ -closure of  $G \subset \text{Diff}^\omega(M)$  if  $\tilde{U}$  can be chosen so as to satisfy the following condition:

- (c): For every  $t \in \mathbb{R}$  and  $\tilde{V} \subset \tilde{U}$  such that  $\phi^s$  is defined on  $\tilde{V}$  for all  $s \in [0, t]$ , there exists a sequence of elements  $g_i \in G$  possessing holomorphic extensions  $\tilde{g}_i$  to  $\tilde{V}$  and such that these extensions  $\tilde{g}_i$  converge uniformly on  $\tilde{V}$  to  $\phi^t : \tilde{V} \rightarrow \phi^t(\tilde{V}) \subset \tilde{M}$ .

To abridge notations, the sequence  $\{g_i\} \subset G$  is said to  $C^\omega$ -approximate  $\phi^t$  when the condition (c) is verified (the phrase  $X$  is in the  $C^\omega$ -closure of  $G$  will also be used). Similarly, when a statement is valid for all  $C^r, C^\infty$  and  $C^\omega$  closures of a group  $G$ , then we shall simply talk about the closure of  $G$  which will be denoted by  $\overline{G}$ .

Obviously the notion of belonging to the  $(C^r, C^\infty, C^\omega)$  closure of  $G$  is relative to the domain of definition of the vector field in question: this will implicitly be understood whenever no misunderstanding is possible. Let us also point out that a similar notion can be defined for diffeomorphisms: a diffeomorphism  $\phi : U \rightarrow \phi(U) \subset M$  is said to be in the  $(C^r, C^\infty)$  closure of  $G$  if it can be approximated in the  $(C^r, C^\infty)$  topology by the restriction to  $U$  of actual elements of  $G$ . To be in the  $C^\omega$ -closure of  $G$ , the diffeomorphism  $\phi : U \rightarrow \phi(U) \subset M$  needs to have a holomorphic extension to a neighborhood  $\tilde{U} \subset \tilde{M}$  of its domain of definition where it is, in addition, a uniform limit of (holomorphic extensions of) elements in  $G$ .

It is clear from the above definition that vector fields in  $\overline{G}$  form a *pseudo-Lie algebra* in the sense that new vector fields obtained out of vector fields in  $\overline{G}$  by means of linear combinations (with constant coefficients) and of Lie brackets still belong to the closure of  $G$  relative to the open set where they are defined. Furthermore it is equally clear that this pseudo-Lie algebra is closed: if  $\{X_i\} \subset \overline{G}$  is a sequence of vector fields defined on some common domain  $U$  and converging (in the corresponding topology) to a vector field  $X_\infty$ , then  $X_\infty$  is automatically contained in  $\overline{G}$ . Finally,  $G$  acts on its pseudo-Lie algebra by pull-backs. The pull-back of a vector field  $X \in \overline{G}$  relative to the domain of definition  $U_X$  of  $X$  by a diffeomorphism  $f$  in  $G$  lies in  $\overline{G}$  relative to its domain of definition  $f^{-1}(U_X)$ . All these remarks apply to local diffeomorphisms in  $\overline{G}$  as well.

Vector fields in the closure of  $G$  constitute a powerful tool to access the dynamics of  $G$ . Naturally, when  $G$  is contained in some (finite dimensional) Lie group acting on  $M$ , non-trivial vector fields in the closure of  $G$  that, in addition, are globally defined on  $M$ , always exist. In fact, being non-discrete, the closure of  $\overline{G}$  inside the Lie group in question is itself a Lie group with non-trivial Lie algebra. The mentioned vector fields are therefore nothing but the image of vector in the Lie algebra in question by the induced representation in the space of vector fields on  $M$ . This situation where  $G$  is contained in a Lie group acting on  $M$  will often be called “linear” as opposed to the “non-linear” situation where  $G$  is simply a finitely generated subgroup of  $\text{Diff}^\omega(M)$ . In the “non-linear” setting, however, the existence of these vector fields is in general hard to establish.

Let us begin by pointing out the type of information required to construct non-identically zero vector fields in the closure of a group. For this, consider an open set  $U \subset M$  which may be identified to a ball about the origin in  $\mathbb{R}^n$ . Suppose that a group  $G$  of diffeomorphisms of  $M$  contains a sequence of elements  $\{h_i\}$  verifying the following conditions.

- The restrictions to  $U$  of the diffeomorphisms  $h_i$  form a sequence  $\{h_i|_U\}$  of one-to-one maps converging  $C^\infty$  to the identity. Besides  $h_i|_U \neq \text{id}$  for every  $i \in \mathbb{N}$ .
- For some  $r \geq 1$  in  $\mathbb{N}$ , there is a uniform constant  $\text{Const}$  such that

$$\|h_i - \text{id}\|_{r,U} < \text{Const} \|h_i - \text{id}\|_{0,U}$$

where, for  $k \in \mathbb{N}$ , the bars  $\|\cdot\|_{k,U}$  stand for the  $C^k$ -norm on  $U$ .

Then we have:

**Proposition 3.1.** *Assume that  $G$  contains a sequence of elements satisfying the above conditions for some  $r \geq 1$ . Then there exists a non-identically zero vector field  $X$  defined on  $U$  and contained in the  $C^{r-1}$ -closure of  $G$ .*

*Proof.* For every  $i \in \mathbb{N}$ , let  $C_i$  denote the  $C^0$ -norm of  $h_i - \text{id}$  on  $U$ , i.e.  $C_i = \|h_i - \text{id}\|_{0,U}$ . Then consider the vector field  $X_i$  defined on  $U$  by

$$(1) \quad X_i(x) = \frac{1}{C_i} \cdot (h_i(x) - x),$$



where  $x = (x_1, \dots, x_n)$  and where  $h_i(x) - x$  stands for the vector going from the point  $x$  to the point  $h_i(x)$ .

Next note that the  $C^r$ -norm on  $U$  of the vector fields  $X_i$  is uniformly bounded by  $\text{Const}$ . Thus, by applying Ascoli-Arzelà theorem, we can suppose without loss of generality that the sequence of vector fields  $X_i$  converge in the  $C^{r-1}$ -topology to a  $C^{r-1}$  vector field  $X_\infty$  defined on  $U$ . The definition of  $C_i$  immediately implies that  $X_\infty$  is not identically zero since  $\|X_i(x)\|_{0,U} = 1$  for every  $i \in \mathbb{N}$ .

Finally let us prove that the local flow  $\phi^t$  of  $X_\infty$  can be  $C^{r-1}$ -approximated by (suitable restrictions of) elements in  $G$ . For this, consider an open set  $U_0$  compactly contained in  $U$  and  $t_0 \in \mathbb{R}$  so that  $\phi^t : U_0 \rightarrow U \subset \mathbb{R}^n$  is defined whenever  $0 \leq t \leq t_0$ . It is then a straightforward application of Euler's polygonal method to check that the (restriction to  $U_0$ ) of the elements in  $G$  given by  $h_i^{[tt_0/C_i]}$  converges on  $U_0$  to  $\phi_X^t$  in the  $C^{r-1}$ -topology (where the brackets  $[\cdot]$  stand for the integral part). The proof of the proposition is over.  $\square$

Note that the statement above *cannot* directly be adapted to handle the  $C^\infty$ -case. An analytic version of it, however, can easily be obtained.

**Proposition 3.2.** *Assume that  $G \subset \text{Diff}^\omega(M)$  contains a sequence  $\{h_i\}$  of elements satisfying the following conditions:*

- *The elements  $\{h_i\}$  possess holomorphic extensions  $\{\tilde{h}_i\}$  to a (fixed) neighborhood  $\tilde{U} \subset \tilde{M}$  of  $U \subset M \subset \tilde{M}$ . Moreover  $h_i \neq \text{id}$  for every  $i \in \mathbb{N}$ .*
- *Denoting by  $\|\cdot\|_{\tilde{U}}$  the  $C^0$ -norm of a function defined on  $\tilde{U}$ , the sequence  $\{h_i\}$  is such that  $\|\tilde{h}_i - \text{id}\|_{\tilde{U}}$  converges to zero as  $i \rightarrow \infty$ .*
- *There exists  $V \subset U$  and a neighborhood  $\tilde{V} \subset \tilde{M}$  compactly contained in  $\tilde{U}$  along with a uniform constant  $C$  such that*

$$\|\tilde{h}_i - \text{id}\|_{\tilde{U}} < \text{Const} \|\tilde{h}_i - \text{id}\|_{\tilde{V}},$$

*for every  $i \in \mathbb{N}$ .*

*Then there is a non-identically zero analytic vector field  $X$  defined on  $V$  and contained in the  $C^\omega$ -closure of  $G$ .*

*Proof.* The proof is an immediate adaptation of the proof of Proposition 3.1. It suffices to replace Ascoli-Arzelà theorem by Montel theorem.  $\square$

Before continuing our discussion, let us state a useful elementary lemma. Fixed  $d \in \mathbb{N}$ , denote by  $\text{Pol}(d, n)$  the space of maps from  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  whose components are polynomials of degree at most  $d$  in the multi-variable  $x = (x_1, \dots, x_n)$ .

**Lemma 3.3.** *Let  $\{P_j\} \subset \text{Pol}(d, n)$  be a sequence of polynomials converging to the identity, with  $P_j \neq \text{id}$  for every  $j \in \mathbb{N}$ . Let  $V \subset \mathbb{R}^n$  be an open set and suppose we are given  $r \in \mathbb{N}$ . Then there is a constant  $C = C(d, n, V, r)$  such that*

$$(2) \quad \|P_j - \text{id}\|_{r,V} < C_r \|P_j - \text{id}\|_{0,V}$$

*uniformly on  $j$ . Similarly if  $\tilde{V} \in \mathbb{C}^n$  is a compact neighborhood of  $V$  and we consider the polynomials  $P_j$  as defined on  $\mathbb{C}^n$ , then there is a constant  $C$  such that*

$$(3) \quad \|P_j - \text{id}\|_{\tilde{V}} < C_r \|P_j - \text{id}\|_{0,V}$$

*uniformly on  $j$ .*

*Proof.* By virtue of Cauchy formula, the second assertion in the statement implies the first one. Thus it suffices to prove Estimate (3). Since  $\{P_j\}$  has bounded degree and it converges to the identity in the  $C^0$ -topology over the open set  $U$ , it is clear that the coefficients of  $P_j$  converge to the corresponding coefficients of the identity map  $x = (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n)$ .

Suppose that a small  $\delta > 0$  is fixed. Modulo multiplying each  $P_j - \text{id}$  by some constant  $c(j) \in \mathbb{R}$ , we may assume that, for every  $j$ , all the coefficients of  $P_j - \text{id}$  have absolute value bounded by  $\delta$  and, in addition, that at least one of these coefficients has absolute value exactly equal to  $\delta$ . Furthermore, if  $\delta > 0$  is chosen sufficiently small, then the norm  $\|P_j - \text{id}\|_{\tilde{V}}$  is still bounded by 1 (recall that  $\tilde{V}$  is compact). Now suppose for a contradiction that the statement is false. Since  $\|P_j - \text{id}\|_{\tilde{V}} \leq 1$ , modulo passing to a subsequence, we can suppose that  $\|P_j - \text{id}\|_{0,V} \rightarrow 0$ , otherwise there is nothing to be proved. Nonetheless the space  $\text{Pol}(d, n)$  is locally compact since it is modeled by some finite dimensional Euclidean space. Furthermore all the coefficients of  $P_j - \text{id}$  have absolute value bounded by  $\delta$  so that, again modulo passing to a subsequence, we can assume that the sequence  $\{P_j - \text{id}\}$  converges towards a polynomial map  $Q_\infty$ . The polynomial  $Q_\infty$  is not identically zero since at least one of its coefficients has absolute value equal to  $\delta > 0$ . A contradiction then arises from observing that  $\|Q_\infty\|_{0,V}$  must be zero since it is the limit of the sequence  $\{\|P_j - \text{id}\|_{0,V}\}$  which converges to zero. The lemma is proved.  $\square$

For the rest of this section, we shall exclusively deal with groups of diffeomorphisms of  $M$  that are smooth or real analytic, the more technical adaptations related to the  $C^r$ -case will be left to the reader.

Consider a (smooth or real analytic) diffeomorphism  $F$  of  $M$  fixing a point  $p \in U$ . In local coordinates, we can think of  $F$  as a local diffeomorphism of  $\mathbb{R}^n$  fixing the origin. Suppose that the point  $p$  (identified to the origin) is a contracting fixed point of  $F$ . Consider also a sequence of elements  $\{g_i\} \subset G$  whose restrictions  $g_{i,U}$  to  $U$  are different from the identity for every  $i \in \mathbb{N}$ . The sequence  $\{g_{i,U}\}$  is supposed to satisfy *one of the following assumptions*.

SC: The sequence  $\{g_{i,U}\}$  converges to the identity on  $U$  in the  $C^\infty$ -topology.

HC: There is a neighborhood  $\tilde{U} \subset \tilde{M}$  of  $U$  where all the maps  $g_{i,U}$  have a holomorphic extension denoted by  $\tilde{g}_{i,U}$ . Moreover the sequence  $\{\tilde{g}_{i,U}\}$  converges uniformly to the identity on  $\tilde{U}$ .

The rest of this section is devoted to the proof of a couple of propositions, namely:

**Proposition 3.4.** *For  $U$  as above, suppose that the group  $G$  contains an element  $F$  which has a non-resonant contracting fixed point  $p \in U$ . Suppose, in addition, that  $G$  contains a sequence of elements  $\{g_i\}$  satisfying condition (SC). Then, given  $r \in \mathbb{N}$  and modulo reducing  $U$ , there exists a non-identically zero vector field  $X$  defined on  $U$  and contained in the  $C^{r-1}$ -closure of  $G$ . Besides  $X$  has the same regularity as  $F$ , i.e.  $X$  is smooth/analytic provided that  $F$  is so.*

This proposition admits the following “analytic” analogue:

**Proposition 3.5.** *For  $U$  as above, suppose that the group  $G$  contains an element  $F$  which has a non-resonant contracting fixed point  $p \in U$ . Suppose, in addition, that  $G$  contains a sequence of elements  $\{g_i\}$  satisfying condition (HC). Then, modulo reducing  $U$ , there exists a non-identically zero real analytic  $X$  defined on  $U$  and contained in the  $C^\omega$ -closure of  $G$ .*

**Remark 3.6.** Before approaching the proofs of these propositions, let us point out that the assumption concerning the non-resonant character of the contracting fixed point of  $F$  is not indispensable for the corresponding statements to hold. To substantiate this claim, observe

that a contracting fixed point can have only finitely many resonance modes. Thus, as long as  $F$  is smooth (resp. analytic),  $F$  is smoothly (analytically) conjugate to a polynomial normal form, called Poincaré-Dulac normal form, cf. [Ar-II]. The use of this Poincaré-Dulac normal form enables one to check that the fundamental estimates used below, namely Estimate (6) and Estimate (7), still hold in the corresponding context. In other words, the assumption about non-resonance is only intended to simplify the calculations since  $F$  becomes a diagonal matrix in appropriate coordinates.

To prove Proposition 3.4, the central point is to construct a sequence  $\{h_i\} \subset G$  fulfilling the conditions of Proposition 3.1, for every *a priori* fixed  $r \in \mathbb{N}$ . Similarly the sequence  $\{h_i\} \subset G$  is supposed to satisfy the conditions of Proposition 3.2 in the case of Proposition 3.5. Let us first work out the proof of Proposition 3.4.

Let  $U$  be identified to the unit ball of  $\mathbb{R}^n$  and let  $p$  be identified to the origin. Consider the sequence  $\{g_{i,U}\} \subset G$ ,  $g_{i,U} \neq \text{id}$  for every  $i \in \mathbb{N}$ , converging  $C^\infty$  to the identity and obtained as restrictions of elements in  $G$ . To abridge notations, this sequence will simply be denoted by  $\{g_i\}$  since no misunderstanding is possible. Denote by  $P_i^r(x)$  the Taylor polynomial  $P_i^r(x) = g_i(0) + g_i^{(1)}(0)x + \cdots + g_i^{(r)}(0)x^r/r!$ . We have

$$(4) \quad \|g_i(x) - P_i^r(x)\| \leq \frac{1}{(r+1)!} \sup_{x \in U} \|g_i^{(r+1)}(x)\| \cdot \|x\|^{r+1}.$$

Since  $\{g_i\}$  converges  $C^\infty$  to the identity, it follows that  $\sup_{x \in U} \|g_i^{(k)}(x)\|$  converges to zero for  $k \neq 1$  while, for  $k = 1$ ,  $\sup_{x \in U} \|g_i^{(1)}(x)\|$  converges to the identity. If the sequence  $\{g_i\}$  satisfies the assumptions of Proposition 3.1, then there is a non-identically zero vector field in the  $C^{r-1}$ -closure of  $G$ , though it will not be clear that  $X$  can be chosen as a real analytic vector field. In general, we proceed as follows.

*Proof of Proposition 3.4.* Recall that  $U$  is identified to a neighborhood of the origin in  $\mathbb{R}^n$ . Modulo working over  $\mathbb{C}$ ,  $F$  becomes identified to the homothety  $(x_1, \dots, x_n) \mapsto (\lambda_1 x_1, \dots, \lambda_n x_n)$  where the eigenvalues  $\lambda_i$  of  $D_0 F$  verify  $\sup_{i=1, \dots, n} |\lambda_i| < 1$  (alternatively it is possible to work with the “block-diagonal” form of  $D_0 F$  involving both real and purely imaginary parts of the eigenvalues).

The next step consists of constructing a sequence  $\{h_i\} \subset G$  verifying the assumptions of Proposition 3.1. The sequence  $\{h_i\}$  will be constructed by means of the sequence  $\{g_i\}$  by setting  $h_i = F^{-k(i)} \circ g_i \circ F^{k(i)}$  for a suitable sequence  $\{k(i)\} \subset \mathbb{N}$  to be defined below. To prove that a suitable choice of integers  $k(i)$  leads to a sequence  $h_i$  verifying the indicated conditions, let us first permute the coordinates so as to have  $|\lambda_1| < \cdots < |\lambda_n| < 1$ . In particular, the subset of  $\mathbb{R}$  consisting of the absolute values of all possible combinations having the form  $\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} / \lambda_l$ ,  $\alpha_1, \dots, \alpha_n \in \mathbb{N}$  and  $l \in \{1, \dots, n\}$  admits a supremum. Besides this supremum is attained by finitely many elements.

Choose  $N$  large enough to have  $|\lambda_n|^N < |\lambda_1|$ . Without loss of generality, we can assume that  $r > N$ . Finally, setting  $\Lambda^s.x = (\lambda_1^s x_1, \dots, \lambda_n^s x_n)$ , it follows that

$$h_{i,k} = [D_0 F]^{-k} \cdot g_i \circ F^k = [D_0 F]^{-k} \cdot \left[ \sum_{j=0}^r g_i^{(j)}(0) \Lambda^{kj}.x + R_{i,k}(x) \right]$$

for every  $k \in \mathbb{N}$ , where

$$(5) \quad [D_0 F]^{-k} \cdot R_{i,k}(x) \leq \frac{1}{(r+1)!} [D_0 F]^{-k} \sup_{x \in U} \|g_i^{(r+1)}(x)\| \cdot \|\Lambda^{k(r+1)}.x\| < \text{Const} \sup_{x \in U} \|g_i^{(r+1)}(x)\|,$$

for a uniform constant  $\text{Const}$  (recall that  $|\lambda_n|^N < |\lambda_1|$ ). Now, note that  $\sup_{x \in U} \|g_i^{(r+1)}(x)\|$  converges to zero with  $i$  since  $g_i$  converges in the  $C^\infty$ -topology to the identity. By using, for example, the integral representation for the Taylor remainder term, it is straightforward to obtain similar estimates for derivatives so as to ensure that the  $C^r$ -norm of  $[D_0 F]^{-k} \cdot R_{i,k}(x)$  on  $U$  converges to zero with  $i$  and independently of  $k$ . Fix then a decreasing sequence  $\{\delta_i\}$  converging to 0 such that

$$(6) \quad \|[D_0 F]^{-k} \cdot R_{i,k}(x)\|_{r,U} < \delta_i$$

for every  $i, k \in \mathbb{N}$ . Next, consider the sequence of polynomial maps  $P_{i,k}$  given by  $P_{i,k} = [D_0 F]^{-k} \cdot [\sum_{j=0}^r g_i^{(j)}(0) \Lambda^{kj} \cdot x]$ . These polynomials are elements of  $\text{Pol}(r, n)$  converging to the identity map so that Lemma 3.3 yields a constant  $C$  verifying

$$(7) \quad \|P_{i,k} - \text{id}\|_{r,U} < C \|P_{i,k} - \text{id}\|_{0,U}.$$

For the convenience of the reader, the argument will now be split into two cases.

*Case 1.* Modulo passing to a subsequence, suppose that  $g_i(0) \neq 0$  for every  $i$ .

In this case, for every  $i$  fixed, there is a smallest positive integer  $K_i$  such that

$$(8) \quad \sup_{x \in U} \|[D_0 F]^{-k(i)} \cdot [\sum_{j=0}^r g_i^{(j)}(0) \Lambda^{k(i)j} \cdot x - \text{id}]\| > \max\{10\delta_i, 10C\delta_i\}$$

where  $C$  is the constant involved in Estimate (7). The existence of  $K_i$  follows easily from the fact that  $\|g_i(0) - 0\| > 0$  so that  $\lim_{k \rightarrow \infty} \|[D_0 F]^{-k} \cdot (g_i(0) - 0)\| = \infty$ . Now set  $k(i) = K_i$  so as to define  $h_i = h_{i,k(i)} = h_{i,K_i}$ . We need to check that the resulting sequence  $h_i = F^{-k(i)} \circ g_i \circ F^{k(i)}$  satisfies the assumptions of Proposition 3.1. The first step is to show that this sequence converges to the identity (on  $U$ ) in the  $C^\infty$ -topology. This goes as follows. Recall that

$$h_i = [D_0 F]^{-k(i)} \cdot g_i \circ F^{k(i)} = [D_0 F]^{-k(i)} \cdot [\sum_{j=0}^r g_i^{(j)}(0) \Lambda^{k(i)j} \cdot x + R_{i,k(i)}(x)].$$

The function  $[D_0 F]^{-k(i)} \cdot R_{i,k(i)}$  clearly converges in the  $C^\infty$ -topology to the identically zero function as it follows from using, for example, the integral form of Taylor remainder term (the  $C^0$ -norm was explicitly estimated in Formula (5)). Thus we only need to show that the truncation  $J^r h_i$  given by

$$J^r h_i = [D_0 F]^{-k(i)} \cdot [\sum_{j=0}^r g_i^{(j)}(0) \Lambda^{k(i)j} \cdot x]$$

converges to the identity in the  $C^\infty$ -topology. For this recall that  $k(i)$  was defined as the smallest positive integer verifying Estimate (8). In turn, this implies that

$$\sup_{x \in U} \|[D_0 F]^{-k(i)-1} \cdot [\sum_{j=0}^r g_i^{(j)}(0) \Lambda^{(k(i)-1)j} \cdot x - \text{id}]\| \leq \max\{10\delta_i, 10C\delta_i\}.$$

In particular,  $\sup_{x \in U} \|J^r h_i - \text{id}\|$  is bounded by  $\lambda_1^{-1} \max\{10\delta_i, 10C\delta_i\}$ . Since  $\delta_i \rightarrow 0$ , it follows that  $\{\|J^r h_i - \text{id}\|_{0,r}\}$  converges to zero as well. By virtue of Lemma 3.3, we conclude that  $\{\|J^r h_i - \text{id}\|_{0,r}\}$  converges to zero in the  $C^k$ -topology for every  $k$  what is equivalent to saying that it converges to zero in the  $C^\infty$ -topology.

Finally to ensure that the second condition of Proposition 3.1 is also verified, we proceed as follows. First note that we have

$$\begin{aligned}
 \|h_i - \text{id}\|_{0,U} &= \|J^r h_i - \text{id} + [D_0 F]^{-k(i)} \cdot R_{i,k(i)}(x)\|_{0,U} \\
 &\geq \|J^r h_i - \text{id}\|_{0,U} - \|[D_0 F]^{-k(i)} \cdot R_{i,k(i)}(x)\|_{0,U} \\
 (9) \qquad \qquad &\geq \max\{9\delta_i, 9C\delta_i\},
 \end{aligned}$$

where estimates (8) and (6) were used.

On the other hand, we also have

$$\begin{aligned}
 \|h_i - \text{id}\|_{r,U} &\leq \|J^r h_i - \text{id}\|_{r,U} + \|[D_0 F]^{-k(i)} \cdot R_{i,k(i)}(x)\|_{r,U} \\
 &\leq C\|J^r h_i - \text{id}\|_{0,U} + \delta_i \\
 (10) \qquad \qquad &\leq C\max\{10\delta_i, C\delta_i\} + \delta_i,
 \end{aligned}$$

where estimates (8) and (7) were used. The statement now follows from comparing Estimates (9) and (10).

Summarizing, it was shown above the existence of a sequence  $\{h_i\} \subset G$  satisfying the conditions of Proposition 3.1. Therefore there exists a non-identically zero vector field  $X$  of class  $C^{r-1}$  lying in the  $C^{r-1}$ -closure of  $G$ . It remains to show that, in fact,  $X$  can be chosen smooth/analytic according to the regularity of  $F$ . However, to check this assertion, it suffices to revisit the sequence  $\{h_i\}$  constructed above. Indeed, it follows from the previous construction that  $h_i$  can be written as  $h_i = \bar{h}_i + R_i$  where  $\bar{h}_i, R_i$  satisfy the following conditions.

- $\bar{h}_i$  is a polynomial whose degree is uniformly bounded (independently of  $i$ ). Moreover all the coefficients of  $\bar{h}_i - \text{id}$  have absolute value bounded by  $\delta > 0$ .
- At least one of the coefficients of  $\bar{h}_i - \text{id}$  has absolute value exactly equal to  $\delta$ .
- The sequence of quotients defined by  $\|R_i\|_{r,U}/\|\bar{h}_i\|_{0,U}$  converges to zero.

Now recall that  $X$  is the  $C^{r-1}$ -limit (and hence the  $C^0$ -limit) of the sequence of vector fields  $X_i$  defined in Equation (1). However, the fact that the quotients  $\|R_i\|_{r,U}/\|\bar{h}_i\|_{0,U}$  converge to zero implies that the sequence  $X_i$  has the same  $C^0$ -limit as the sequence  $Y_i$  given by

$$Y_i(x) = \frac{1}{\|\bar{h}_i - \text{id}\|_{0,U}} \cdot (\bar{h}_i(x) - x).$$

It is clear that the  $C^0$ -limit of  $\{Y_i\}$  is a polynomial vector field and, the  $C^0$ -limit being unique, it must coincide with  $X$ . Therefore  $X$  is actually polynomial in the coordinate where  $F$  is linearizable. As already seen, this coordinate is smooth/analytic depending on whether  $F$  is smooth/analytic. The proposition then follows in Case 1.

Now we need to consider the case where  $g_i(0) = 0$  for every  $i$ . Concerning the previous discussion, the only additional difficulty this may pose lies in the fact that a (minimal) positive integer  $K_i$  fulfilling Estimate (8) may fail to exist. Indeed, it may even happen that the sequence of maps  $\{F^{-k} \circ g_i \circ F^k\}_k$ , indexed by  $k$  and having  $i$  fixed, converges to the identity. Here two different cases have to be considered.

*Case 2.* Suppose that  $K_i$  as before do not exist (for all but finitely many  $i$ ). In particular  $g_i(0) = 0$  for  $i$  large enough.

Modulo discarding finitely many values of  $i$ , we assume that  $K_i$  do not exist for every  $i$ . Suppose first the existence of some  $i_0$  for which the sequence of maps  $H_k = F^{-k} \circ g_{i_0} \circ F^k$  (indexed by  $k$ ) converges to the identity. This case happens for example if there is  $i_0$  such that  $g_{i_0}$  is tangent to the identity to an order  $N$  satisfying  $|\lambda_n|^N < |\lambda_1|$ . Nonetheless, for  $k$  large, the value of  $\|F^{-k} \circ g_{i_0} \circ F^k - \text{id}\|_{r,U}$  is mostly concentrated over a polynomial part of the Taylor

series of  $F^{-k} \circ g_{i_0} \circ F^k$ . This polynomial part has degree fixed (i.e. uniformly bounded on  $k$ ) since it is determined solely by the Taylor series of  $g_{i_0}$ . Thus Lemma 3.3 can be employed to ensure that the sequence  $H_k$  satisfies the conditions of Proposition 3.1 and hence there is a non-identically zero vector field in the  $C^{r-1}$ -closure of  $G$ . To check that  $X$  can again be chosen smooth/analytic according to the regularity of  $F$ , proceed as before to conclude that each  $H_k$  can be represented under the form  $\overline{H}_k + R_k$  where the following holds:

- $\overline{H}_k$  is a polynomial of uniformly bounded degree.
- The sequence of quotients defined by  $\|R_i\|_{r,U}/\|\overline{H}_i\|_{0,U}$  converge to zero.

In view of this, the previous argument allows to conclude again that  $X$  is polynomial in the coordinates where  $F$  is linearizable. The proposition follows in the present case.

Suppose now that for no value  $i_0$  of  $i$  the sequence (indexed by  $k$ ) of maps  $F^{-k} \circ g_i \circ F^k$  converges to the identity. In particular, there is a uniform  $N$  such that no  $g_i$  is tangent to the identity to order  $N$ . Now, for  $i$  fixed, consider again the sequence  $h_{i,k} = F^{-k} \circ g_i \circ F^k$ . As  $k$  becomes large, the value of  $\|F^{-k} \circ g_i \circ F^k - \text{id}\|_{r,U}$  is mostly concentrated over the polynomial part of degree  $N$  of the Taylor series of  $F^{-k} \circ g_i \circ F^k$ . Hence we can choose a convenient sequence of pairs  $(i, k(i))$  leading to maps  $h_{i,k(i)}$  converging to the identity and such that the value of  $\|F^{-k(i)} \circ g_i \circ F^{k(i)} - \text{id}\|_{r,U}$  becomes more and more concentrated over the degree- $N$  Taylor polynomial of  $F^{-k(i)} \circ g_i \circ F^{k(i)} - \text{id}$ . Now the combination of Lemma 3.3 and Proposition 3.1 yields again the existence of  $X$  in the  $C^{r-1}$ -closure of  $G$ . The argument used in the preceding cases can similarly be repeated to show that  $X$  is smooth/analytic depending only on the regularity of  $F$ . The proposition is proved.  $\square$

Let us close this section with the proof of Proposition 3.5.

*Proof of Proposition 3.5.* The argument is analogous though much simpler than the one employed in the proof of Proposition 3.4. In analytic coordinates,  $F$  is again represented by the homothety  $(x_1, \dots, x_n) \mapsto (\lambda_1 x_1, \dots, \lambda_n x_n)$ . Next, we consider maps  $h_{i,k}$  of the form  $h_{i,k} = F^{-k} \circ g_i \circ F^k$ . Besides, owing to HC-assumption, each  $g_i$  is represented by its Taylor series (based at the origin) on a fixed poly-disc about the origin. Since the effect of the conjugation by  $F^k$  decreases the terms of higher order of the Taylor series of  $g_i$  faster than those of lower orders, the previous scheme allows us to choose a sequence  $k(i)$  such that the corresponding maps  $h_i = F^{-k(i)} \circ g_i \circ F^{k(i)}$  satisfy the conditions of Proposition 3.2. Thus there is a non-identically zero analytic vector field  $X$  contained in the  $C^\omega$ -closure of  $G$ . The proposition is established.  $\square$

#### 4. QUASI-INVARIANT MEASURES AND PROOF OF THEOREM A

From now on, all groups of diffeomorphisms  $G$  are supposed to consist of analytic diffeomorphisms, i.e.  $G \subset \text{Diff}^\omega(M)$ . Furthermore, *every local vector field supposed to be contained in the closure of  $G$  is understood to be non-identically zero*. Unless otherwise stated, these vector fields are also real analytic. On the other hand, the group  $G$  is supposed to be non-discrete but only in the  $C^\infty$ -topology. In other words, we shall deal with groups  $G \subset \text{Diff}^\omega(M)$  satisfying conditions (1), (2) and (3) of Section 2.

This section consists of three paragraphs whose contents are as follows. In the first paragraph, ideas used in Section 3 will further be developed to yield vector fields in the  $C^\infty$ -closure of  $G$ . In the second paragraph, we shall extend to higher dimensions a proposition claiming that the class of the Lebesgue measure is the only (class of) quasi-invariant measure for the affine group of the line. A proof for the original proposition can be found in [R-5] and, in the mentioned

paragraph, the corresponding argument will significantly be generalized to higher dimensions. Here it may be noted that the main difficulty involved in this generalization is overcome by the use of a famous result due to Oxtobi and Ulam. Finally, in the last paragraph, all the previous information will be brought together to prove Theorem A and to deduce Corollaries B and C.

**4.1. Smooth vector fields in the closure of a group.** To begin with, let  $F$  denote an analytic Morse-Smale diffeomorphism of  $M$ . By definition, the non-wandering set  $\Omega(F)$  of  $F$  must be finite and thus it must coincide with the set of periodic points  $\text{Per}(F)$  of  $F$ . Moreover, every periodic point  $p$  of  $F$  is hyperbolic with stable and unstable manifolds denoted respectively by  $W^s(p)$  and  $W^u(p)$ . It is still part of the definition of a Morse-Smale diffeomorphism that, for every pair of periodic points  $p, q$  the stable manifold  $W^s(p)$  of  $p$  and the unstable manifold  $W^u(q)$  of  $q$  are in general position. Modulo passing to a finite power of  $F$ , the diffeomorphism  $F$  possesses both *contracting* and *expanding* fixed points. Note that an expanding fixed point for  $F$  is contracting for  $F^{-1}$  and conversely. Let  $p_1, \dots, p_l$  (resp.  $q_1, \dots, q_s$ ) be the contracting fixed points of  $F$  (resp.  $F^{-1}$ ). Denote by  $\text{Bas}_F(p_i)$  (resp.  $\text{Bas}_{F^{-1}}(q_j)$ ) the *basin of attraction* of  $p_i$  (resp.  $q_j$ ), namely the set of points  $x \in M$  such that  $\lim_{k \rightarrow \infty} F^k(x) = p_i$  (resp.  $\lim_{k \rightarrow \infty} F^{-k}(x) = q_j$ ). Now we have the following:

**Lemma 4.1.** *Suppose that  $G \subset \text{Diff}^\omega(M)$  satisfies conditions (1), (2) and (3) of Section 2. Then the  $G$ -orbit of every point  $p$  accumulates at one of the points  $p_1, \dots, p_l, q_1, \dots, q_s$ .*

*Proof.* Consider the union

$$W = \bigcup_{i=1}^l \text{Bas}_F(p_i) \cup \bigcup_{j=1}^s \text{Bas}_{F^{-1}}(q_j).$$

The set  $W \subset M$  is clearly open. To prove the claim, it suffices to check that the  $G$ -orbit of every point in  $M$  intersects  $W$ . For this, note that the complement  $M \setminus W$  of  $W$  in  $M$  consists of those points  $x \in M$  for which there are fixed points  $P$  and  $Q$  of  $F$  satisfying the following conditions.

- $P$  and  $Q$  are (necessarily hyperbolic) fixed points of  $F$  having saddle type (i.e. the corresponding differential have eigenvalues of absolute value less than 1 and eigenvalues of absolute value greater than 1).
- The point  $x$  belongs to the intersection of  $W^u(P) \cap W^s(Q)$ .

Since  $F$  is real analytic, the stable (resp. unstable) manifolds of all its periodic points are analytic as well. Thus these invariant manifolds are analytic sub-manifolds of  $M$ . They are, besides, properly embedded since the  $\omega$ -limit (resp.  $\alpha$ -limit) of every point in  $M$  must be contained in  $\text{Per}(F)$ . Thus, for every pair of periodic points  $P, Q$ , the transverse intersection  $W^u(P) \cap W^s(Q)$  is a proper analytic subset of  $M$ . In other words, the complement  $M \setminus W$  of  $W$  in  $U$  is a proper analytic subset of  $M$ .

To complete the proof of the lemma, we proceed as follows. Consider an irreducible component  $A_1$  of  $M \setminus W$ . According to condition (3) in Section 2, there exists  $g \in G$  such that  $g(A_1)$  is not contained in  $M \setminus W$ . We then consider the intersection  $g(A_1) \cap (M \setminus W)$  and, if not empty, it yields analytic sets with dimension strictly less than the dimension of  $A_1$ . Since the number of irreducible components of  $M \setminus W$  is finite, we can repeatedly use condition (3) until be able to ensure that the corresponding intersections are empty. The proof of the lemma is over.  $\square$

Since the group  $G$  is non-discrete in the  $C^\infty$ -topology, it would be convenient to have vector fields in the  $C^\infty$ -closure of  $G$  whereas Proposition 3.4 produces analytic vector fields that are

only in the  $C^r$ -closure of  $G$ . Though  $r$  can a priori be fixed, the vector field in question depends on this fixed  $r$  so that we cannot immediately derive the existence of a non-trivial  $C^\infty$ -closure for  $G$  as desired. Note that working with the  $C^\infty$ -closure of  $G$  is needed to coherently define a (pseudo) Lie algebra. Indeed, the commutator of two vector fields in the  $C^r$ -closure of  $G$  is, in principle, only in the  $C^{r-1}$ -closure of  $G$ . This gap in the statement of Proposition 3.4 is however bridged by the theorem below.

**Theorem 4.2.** *Assume that  $G \subset \text{Diff}^\omega(M)$  satisfies conditions (1), (2) and (3) of Section 2. Then there exists a finite open covering  $\{U_k\}_{k=1,\dots,l}$  of  $M$  such that every  $U_k$  is endowed with a nowhere zero analytic vector field  $X_k$  contained in the  $C^\infty$ -closure of  $G$ .*

**Remark 4.3.** If  $G$  satisfies conditions (1), (2) and (3) of Section 2 but it is constituted by smooth diffeomorphisms then the statement of Theorem 4.2 still holds except that the resulting vector fields  $X_k$  are only smooth. The proof of this differentiable version of the statement can immediately be obtained from the argument given below modulo replacing the analytic statements from Section 3 by their analogues in the smooth case.

Owing to the compactness of  $M$ , in order to prove Theorem 4.2 it suffices to show that every point in  $M$  belongs to the domain of definition of a nowhere zero analytic vector field in the  $C^\infty$ -closure of  $G$ . In turn, Lemma 4.1 reduces the latter statement to showing that every contracting fixed point  $p$  of  $F$  lies in the domain of definition of an analytic vector field  $X$  in the  $C^\infty$ -closure of  $G$  and satisfying  $X(p) \neq 0$ . Let then  $p$  as above be fixed. First we have:

**Lemma 4.4.** *Fixed  $r \in \mathbb{N} \cup \{\infty\}$ , there is an analytic vector field  $Y_r$  defined about  $p$  and verifying  $Y_r(p) \neq 0$  that is contained in the  $C^r$ -closure of  $G$ .*

*Proof.* Consider the set  $\mathcal{S} \subset M$  consisting of those points  $q \in M$  verifying the following condition: every analytic vector field  $X$  defined about  $q$  and contained in the  $C^r$ -closure of  $G$  is such that  $X(q) = 0$ . Since the combination of Proposition 3.4 and Lemma 4.1 implies that every point in  $M$  belongs to the domain of definition of a (non-identically zero as always) analytic vector field contained in the  $C^r$ -closure of  $G$ , it follows that  $\mathcal{S}$  is a proper analytic set of  $M$ . Besides,  $\mathcal{S}$  is clearly invariant by  $G$ . Therefore, in view of condition (3), the set  $\mathcal{S}$  must be empty and the lemma follows.  $\square$

As mentioned, Theorem 4.2 amounts to finding an analytic vector field  $X$  in the  $C^\infty$ -closure of  $G$  such that  $X(p) \neq 0$ . The argument provided in Lemma 4.4 (in the case  $r = \infty$ ), in turn, ensures that it suffices to construct a non-identically zero analytic vector field  $X$  defined on a neighborhood of  $p$  and contained in the  $C^\infty$ -closure of  $G$ .

The idea to prove Theorem 4.2 is to construct a single vector field  $X$  out of the vector fields  $Y_r$  for every  $r \in \mathbb{N}$ , that will be in the  $C^r$ -closure of  $G$  for every  $r$ . However, before providing details on this construction, some additional terminology will be needed.

Consider a vector-monomial  $Z$ , i.e. a vector field of the form  $Z = cx_1^{\alpha_1} \dots x_n^{\alpha_n} \partial / \partial x_l$ , with  $\alpha_i \in \mathbb{N}$  for every  $i \in \{1, \dots, n\}$  and  $c \in \mathbb{C}$ . Recall that, in the coordinates  $(x_1, \dots, x_n)$ , we have  $F(x) = F(x_1, \dots, x_n) = (\lambda_1 x_1, \dots, \lambda_n x_n)$  so that  $F^*Z$  is a constant multiple of  $Z$ . Let  $c_F$  be the value of this constant so that  $F^*Z = c_F Z$ . The  $F$ -multiplier  $m(F)$  of  $Z$  is nothing but the absolute value  $|c_F|$  of  $c_F$ . A polynomial vector field is said to be  $F$ -homogeneous if all its vector monomials have the same  $F$ -multiplier. Besides, if  $X = \sum_{l=1}^n \sum_{\alpha_1, \dots, \alpha_n \in \mathbb{N}} c_{\alpha_1, \dots, \alpha_n}^l x_1^{\alpha_1} \dots x_n^{\alpha_n} \partial / \partial x_l$  is an analytic vector field and  $cx_1^{\alpha_1} \dots x_n^{\alpha_n} \partial / \partial x_l$  is a vector-monomial of its Taylor series, then the  $F$ -homogeneous component of  $cx_1^{\alpha_1} \dots x_n^{\alpha_n} \partial / \partial x_l$  is the  $F$ -homogeneous vector field consisting of all vector monomials appearing in the Taylor series of  $F$  and having the same  $F$ -multiplier as  $cx_1^{\alpha_1} \dots x_n^{\alpha_n} \partial / \partial x_l$ . The fact that  $|\lambda_1| < \dots < |\lambda_n| < 1$



ensures that the  $F$ -homogeneous component of every vector-monomial appearing in a Taylor series is necessarily a polynomial vector field.

**Remark 4.5.** Consider the space  $\mathcal{G}$  of (smooth or analytic) vector fields defined on some open set about a point  $q \in M$  and contained in the  $C^\infty$ -closure of  $G$ . This space is clearly a (pseudo) Lie algebra over  $\mathbb{R}$  though it may, in principle, be trivial. In any event, being an algebra over  $\mathbb{R}$ , every real multiple  $cX$  of a vector field  $X \in \mathcal{G}$  also lies in  $\mathcal{G}$ . This is however not the case if  $c$  is allowed to be an arbitrary complex number. Since, in our discussion,  $F$  is diagonalized over  $\mathbb{C}$ , this issue would require us to perform some additional verification whenever multiples of vector fields are considered in the course of this section. These verifications however will systematically be left to the reader since they are straightforward. Alternatively, instead of taking  $F$  diagonal over  $\mathbb{C}$ , the reader may work with the standard “block-diagonal” that avoids the use of complex numbers at the expenses of making the notations slightly cumbersome.

We are now able to prove Theorem 4.2.

*Proof of Theorem 4.2.* According to Proposition 3.4 (and Lemma 4.4), for every  $r \in \mathbb{N}$  there exists an analytic vector field  $Y_r$  in the  $C^r$ -closure of  $G$  and verifying  $Y_r(p) \neq 0$  (so that  $Y_r$  is not identically zero). For  $r$  fixed, let  $Z_r^1, Z_r^2, \dots$  denote the  $F$ -homogeneous components of the (analytic) vector field  $Y_r$  labeled in decreasing order of the corresponding  $F$ -multipliers. In other words, if  $m^j(F)$  stands for the  $F$ -multiplier of  $Z_r^j$ , then  $m^{j_1}(F) > m^{j_2}(F)$  provided that  $j_1 < j_2$ . Now we have:

*Claim 1.* The sequence of polynomial vector fields  $\{Z_r^1\}_{r \in \mathbb{N}}$  has uniformly bounded degree.

*Proof of Claim 1.* Note that the set of all  $F$ -multipliers associated to vector-monomials is fully constituted by values of the form  $|\lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n} / \lambda_l|$ , where  $\alpha_i \in \mathbb{N}$  for every  $i \in \{1, \dots, n\}$  and  $l \in \{1, \dots, n\}$ . Since  $Y_r(0) \neq 0$ ,  $Y_r$  has a homogeneous component of multiplier greater than or equal to  $1/|\lambda_1|$  (recall that  $|\lambda_1| < \dots < |\lambda_n| < 1$ ). Thus there may exist only finitely many values of the form  $|\lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n} / \lambda_l|$  that are greater than  $1/|\lambda_1|$  and these values depend only on  $\lambda_1, \dots, \lambda_n$  (in particular they do not depend on  $r$ ). The claim follows.  $\square$

Let  $d$  denote the degree of the first  $F$ -homogeneous component  $Z_r^1$  of  $Y_r$ . We shall prove:

*Claim 2.* There exists a (non-identically zero) polynomial vector field of degree at most  $d$  which is contained in the  $C^r$ -closure of  $G$ .

*Proof of Claim 2.* Let  $r$  and  $Y_r$  be fixed. Denote by  $m^j(F)$  the  $F$ -multiplier of  $Z_r^j$  for  $j \in \mathbb{N}^*$ . For  $j$  fixed, the  $j^{\text{th}}$ - $F$ -homogeneous component  $Z_r^j$  of  $Y_r$  can be split into a sum  $\sum_{l=1}^{m(j)} Z_r^{j,l}$  of vector monomials. Now consider the sequence  $\{Y_r^k\}_{k \in \mathbb{N}}$  (indexed by  $k$ ) of vector fields defined by  $Y_r^k = (F^k)^* Y_r$ . With the previous notations, we have

$$Y_r^k = (F^k)^* Y_r = [m^1(F)]^k \left( \sum_{l=1}^{m(1)} c^{1,l} Z_r^{1,l} \right) + \sum_{j \geq 2} [m^j(F)]^k \left( \sum_{l=1}^{m(j)} c^{j,l} Z_r^{j,l} \right)$$

where all the coefficients  $c^{1,l}$  and  $c^{j,l}$  are complex numbers lying in the unit circle. Clearly all the vector fields  $Y_r^k$  belong to the  $C^r$ -closure of  $G$ . Since constant multiples of vector fields in the closure of  $G$  also belong to the closure of  $G$ , it follows that the new vector fields  $\bar{Y}_r^k$  given by  $\bar{Y}_r^k = (F^k)^* Y_r / (m^1(F))^k$  belong to the  $C^r$ -closure of  $G$  as well. These latter vector fields are however given by

$$\bar{Y}_r^k = \sum_{l=1}^{m(1)} c^{1,l} Z_r^{1,l} + \sum_{j \geq 2} \left[ \frac{m^j(F)}{m^1(F)} \right]^k \left( \sum_{l=1}^{m(j)} c^{j,l} Z_r^{j,l} \right).$$

In particular, by sending  $k \rightarrow \infty$ , it follows that  $\{\overline{Y}_r^k\}$  forms a compact family whose non-trivial accumulation points in the  $C^\infty$ -topology are all contained in the space of polynomial vector fields with degree at most  $d$ . These accumulation points are not identically zero vector fields since all the vector monomials appearing in  $Z_r^1$  will also appear in these limits after being multiplied by some complex number with absolute value equal to 1. Next note that every converging subsequence of  $\{\overline{Y}_r^k\}$  will converge to its limit (polynomial) vector field not only in the  $C^r$  but also in the  $C^\infty$ -topology since the latter is controlled by the topology of (the finitely many) coefficients. Because the  $C^r$ -closure of  $G$  is naturally closed for the  $C^r$ -topology, and given that  $\overline{Y}_r^k$  belongs to the  $C^r$ -closure of  $G$  for every  $k$ , it follows that all the above mentioned limits belong to the  $C^r$ -closure of  $G$  as well. The claim is proved.  $\square$

Let us now conclude the existence of a non-identically zero analytic vector field defined about  $p \simeq 0 \in \mathbb{R}^n$  and contained in the  $C^\infty$ -closure of  $G$ . For each  $r$ , let  $\overline{Z}_r^1$  be a polynomial vector field belonging to the  $C^r$ -closure of  $G$  and obtained from the construction given in the proof of Claim 2. Consider the sequence (indexed by  $r \in \mathbb{N}$ ) of vector fields  $\overline{Z}_r^1$ . It follows from Claim 1 that all the (polynomial) vector fields in this sequence have degree uniformly bounded by some constant  $N \in \mathbb{N}$ . Moreover, modulo multiplying each  $\overline{Z}_r^1$  by suitable constants, we can assume without loss of generality that the maximum of the norm of the corresponding coefficients is exactly 1. Now, since the space of polynomials with bounded degree is locally compact, we can suppose that the vector fields  $\overline{Z}_r^1$  converge (even in the analytic topology) towards a polynomial vector field  $\overline{Z}_\infty$  which does not vanish identically since at least one of its coefficient has absolute value equal to 1. It remains only to check that  $\overline{Z}_\infty$  lies in the  $C^\infty$ -closure of  $G$ . This is however easy: note that, fixed  $r_0 \in \mathbb{N}$ , the vector fields  $\overline{Z}_r^1$  belong to the  $C^{r_0}$ -closure of  $G$  provided that  $r \geq r_0$ . Since the latter converge in the  $C^\infty$ -topology towards  $\overline{Z}_\infty$ , it follows that  $\overline{Z}_\infty$  belongs to the  $C^{r_0}$ -closure of  $G$ . Since  $r_0$  is arbitrary, we conclude that  $\overline{Z}_\infty$  belongs to the  $C^\infty$ -closure of  $G$  completing the proof of the theorem.  $\square$

In closing this paragraph, let us provide a useful and fairly general lemma. Note that the first intrinsic advantage of considering (analytic) vector fields in the  $C^\infty$ -closure of  $G$ , as opposed to vector fields in the  $C^r$ -closure of  $G$ , lies in the fact that the former clearly constitutes a Lie (pseudo-) algebra. Moreover, letting  $p$  be a fixed contracting point of  $F$  and choosing local coordinates where  $F$  becomes the homothety  $(x_1, \dots, x_n) \mapsto (\lambda_1 x_1, \dots, \lambda_n x_n)$ , we have the following.

**Lemma 4.6.** *If  $X$  is an analytic vector field lying in the  $C^\infty$ -closure of  $G$ , then every  $F$ -homogeneous component of  $X$  belongs to the  $C^\infty$ -closure of  $G$  as well.*

*Proof.* Let  $Z^1, Z^2, \dots$  denote the  $F$ -homogeneous components of  $X$  labeled in terms of decreasing  $F$ -multipliers. The central point is to show that the  $F$ -homogeneous component  $Z^1$  belongs to the  $C^\infty$ -closure of  $G$ . Once this has been established, the lemma follows by inductively considering the vector fields  $X - Z^1$ ,  $X - Z^1 - Z^2$  and so on.

To check that  $Z^1$  lies in the  $C^\infty$ -closure of  $G$ , consider  $Z^1$  written as a finite sum of vector-monomials

$$Z^1 = \sum_{l=1}^N a_{\alpha_1(l), \dots, \alpha_n(l)} x_1^{\alpha_1(l)} \dots x_n^{\alpha_n(l)} \partial / \partial x_{\beta(l)}.$$

For  $l$  fixed, the value of the constant  $c_F$  for which the monomial in question is multiplied when pulled-back by  $F$  is going to be denoted by  $c_l$ . In particular, for every  $l$  we have  $c_l = m^1(F) e^{\theta_l \sqrt{-1}}$  where  $m^1(F)$  is the  $F$ -multiplier of  $Z^1$ . As previously seen, the sequence  $Y_k = (F^k)^* X / m^1(F)$  forms a compact family whose non-trivial limit points are contained in the space of polynomial

vector fields with degree bounded by the degree of  $Z^1$ . Next note that, if all the  $\theta_l$  are rational, then there exists a subsequence of  $\{Y_k\}$  converging towards a constant multiple of  $Z^1$  and the statement follows in this case.

The general case can be treated by induction on the number of vector monomials appearing in  $Z^1$ . To simplify the discussion, suppose that this number is two, i.e.  $Z^1 = Z_1^1 + Z_2^1$  where  $Z_1^1, Z_2^1$  are vector-monomials respectively satisfying  $F^*Z_1^1 = m^1(F)e^{\theta_1\sqrt{-1}}$  and  $F^*Z_2^1 = m^1(F)e^{\theta_2\sqrt{-1}}$ . It is then clear that the sequence of vector fields  $Y_k = (F^k)^*X/[m^1(F)]^ke^{k\theta_1\sqrt{-1}}$  has its (proper) accumulation points contained in the set of vector fields having the form

$$Z_1^1 + e^{k(\theta_2 - \theta_1)\sqrt{-1}}Z_2^1.$$

By taking the difference between two conveniently chosen limits as above, we conclude that  $Z_2^1$  lies in the  $C^\infty$ -closure of  $G$ . In turn, this implies that the same holds for  $Z_1^1$  and hence  $Z^1 = Z_1^1 + Z_2^1$  must belong to the  $C^\infty$ -closure of  $G$  as well. Now a standard induction argument completes the proof of the lemma.  $\square$

**4.2. Extensions of the translation group of  $\mathbb{R}^n$  and quasi-invariant measures.** Let  $\mathbf{T}$  denote the group of translations of  $\mathbb{R}^n$  and consider another  $C^\infty$  (or  $C^\omega$ ) diffeomorphism  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  having a contracting fixed point at the origin. The *free product* between  $\mathbb{Z}$  and  $\mathbf{T}$  is going to be denoted by  $\mathbb{Z}\mathbf{T}$ . The choice of  $f$  enables us to define a representation  $\rho_0 : \mathbb{Z}\mathbf{T} \rightarrow \text{Diff}(\mathbb{R}^n)$  by setting  $\rho_0(1 * 0) = f$  and  $\rho_0(0 * B)$  equal to the translation  $x \mapsto x + B$ . The image of  $\mathbb{Z}\mathbf{T}$  by  $\rho_0$  is going to be denoted by  $\Gamma_0$ . Naturally we have  $f \in \Gamma_0$  and  $\mathbf{T} \subset \Gamma_0$ .

Note that the existence of elements with contracting fixed points in  $\Gamma_0$  easily implies that this group does not possess any invariant sigma-finite measure on  $\mathbb{R}^n$ . Thus it is natural to look for  $\Gamma_0$ -invariant *classes of measures* on  $\mathbb{R}^n$ . These are fully characterized by the proposition below.

**Proposition 4.7.** *Every probability measure on  $\mathbb{R}^n$  that is quasi-invariant by the action  $\Gamma_0$  is absolutely continuous with respect to the Lebesgue measure.*

The usual Lebesgue measure  $\text{Leb}$  on  $\mathbb{R}^n$  is clearly quasi-invariant under the action of  $\Gamma_0$ . Since we are dealing only with quasi-invariant measures, it is useful to consider also another Lebesgue measure  $\text{leb}$  on  $\mathbb{R}^n$  arising from normalizing  $\text{Leb}$  by some Gaussian kernel with radial symmetry so as to have  $\text{leb}(\mathbb{R}^n) = 1$ . Let now  $\mu$  denote an arbitrary probability measure on  $\mathbb{R}^n$  that also happens to be quasi-invariant under the action  $\Gamma_0$ . Naturally  $\mu$  cannot give mass to points since  $\Gamma_0$  is uncountable. Furthermore, being quasi-invariant by  $\Gamma_0$ , the support of  $\mu$  coincides with all of  $\mathbb{R}^n$ . At this point, a deep theorem due to Oxtobi and Ulam [O-U] guarantees the existence of a homeomorphism  $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\text{leb} = H^*\mu$ , i.e. for every Borel set  $\mathcal{B} \subseteq \mathbb{R}^n$  we have

$$(11) \quad \text{leb}(H^{-1}(\mathcal{B})) = \mu(\mathcal{B}).$$

By using the homeomorphism  $H$  a new representation  $\rho_1$  from  $\mathbb{Z}\mathbf{T}$  to  $\text{Homeo}(\mathbb{R}^n)$  can be constructed by letting  $\rho_1 = H^{-1} \circ \rho_0 \circ H$ . The image of  $\mathbb{Z}\mathbf{T}$  by  $\rho_1$  will be denoted by  $\Gamma_1 \subset \text{Homeo}(\mathbb{R}^n)$ . Clearly  $\Gamma_0, \Gamma_1$  are isomorphic as abstract groups. Next, we claim that every element  $\gamma_1 \in \Gamma_1$  is an absolutely continuous homeomorphism of  $\mathbb{R}^n$ . To check the claim, denote by  $\gamma_0$  the element of  $\Gamma_0$  satisfying  $\gamma_1 = H^{-1} \circ \gamma_0 \circ H$ . Suppose then that  $\mathcal{B}$  is a Borel set such that  $\text{leb}(\mathcal{B}) = 0$ . Since  $H$  is invertible,  $\mathcal{B} = H^{-1}(H(\mathcal{B}))$  and thus  $\mu(H(\mathcal{B})) = 0$ . However  $\mu$  is quasi-invariant under  $\Gamma_0$  so that  $\mu[\gamma_0 \circ H(\mathcal{B})] = 0$ . In turn, Equation (11) provides  $\text{leb}[(H^{-1} \circ \gamma_0 \circ H)(\mathcal{B})] = 0$  i.e.  $\text{leb}[\gamma_1(\mathcal{B})] = 0$ . Since  $\gamma_1$  is an arbitrary element of  $\Gamma_1$  the claim follows at once.

Going back to  $\Gamma_0$ , the group of translations  $\mathbf{T}$  is naturally a subgroup of  $\Gamma_0$ . As to  $\Gamma_1$ , the image under  $\rho_1$  of  $0 * \mathbf{T} \subset \mathbb{Z}\mathbf{T}$  yields a commutative and simply transitive group of (absolutely continuous) homeomorphism of  $\mathbb{R}^n$  that is going to be denoted by  $\mathbf{T}^H$ . Note that the action of  $\mathbf{T}^H$  is naturally identified to  $H^{-1} \circ \mathbf{T} \circ H$ . Since  $\mathbf{T}$  preserves the Lebesgue measure  $\text{Leb}$  on  $\mathbb{R}^n$ , the group  $\mathbf{T}^H$  preserves the pull-back  $H^*\text{Leb}$  of  $\text{Leb}$  by  $H$ . In particular  $\mathbf{T}^H$  possesses a sigma-finite invariant measure. Now we state:

**Lemma 4.8.** *There is a sigma-finite measure  $\nu$  on  $\mathbb{R}^n$  satisfying the following conditions:*

- (1)  $\nu$  is invariant under  $\mathbf{T}^H$ .
- (2)  $\nu$  is absolutely continuous with respect to  $\text{Leb}$  (or, equivalently, to  $\text{leb}$ ).

*Proof.* Note that  $\mathbf{T}^H$  is the image of the continuous homomorphism  $\rho : \mathbb{R}^n \rightarrow \text{Homeo}(\mathbb{R}^n)$  given by  $\rho(B)(x) = H^{-1}(H(x) + B)$  since  $H$  conjugates  $\mathbf{T}$  and  $\mathbf{T}^H$ . Besides, we already know that the element  $\rho(B) \in \text{Homeo}(\mathbb{R}^n)$  is absolutely continuous for every  $B \in \mathbb{R}^n$ . Therefore, given  $B \in \mathbb{R}^n$ , the Radon-Nikodym derivative of  $[\rho(B)]^*\text{Leb}$  with respect to  $\text{Leb}$  at the point at  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  can be considered. The resulting function, denoted by  $\text{Det}(\rho(B)) : \mathbb{R}^n \rightarrow \mathbb{R}$ , is clearly nonnegative and belongs to  $L^1_{\text{loc}}$  (where, as usual,  $L^1$ ,  $L^1_{\text{loc}}$  are defined with respect to the Lebesgue measure).

Let us now consider the “evaluation map” at  $z \in \mathbb{R}^n$  defined by  $B \mapsto \text{Det}(\rho(B))(z)$  where  $z \in \mathbb{R}^n$  is fixed. Now we have:

*Claim:* For almost all  $z \in \mathbb{R}$  the evaluation map  $B \mapsto \text{Det}(\rho(B))(z)$  belongs to  $L^1_{\text{loc}}$ .

*Proof of the Claim.* Consider two points  $p = (p_1, \dots, p_n)$  and  $q = (q_1, \dots, q_n)$  of  $\mathbb{R}^n$ . Let  $B = (b_1, \dots, b_n)$  be fixed. The statement would follow from Fubini’s theorem, if we can ensure that the “double integral”

$$\iint_{(z,B) \in \mathbb{R}^n \times \mathbb{R}^n} \text{Det}(\rho(B))(z) dB dz$$

is well-defined. Nonetheless, since  $\text{Det}(\rho(B))$  is nonnegative, the classical Tonelli’s theorem (cf. [Bt], page 118) reduces this proof to showing that the integral

$$\int_{p_1}^{q_1} \dots \int_{p_n}^{q_n} \text{Det}(\rho(B))(z) dz$$

exists for every  $B \in \mathbb{R}^n$ . By construction, the latter integral is nothing but the Lebesgue measure  $\text{Leb}$  of the solid  $H([p_1, q_1] \times \dots \times [p_n, q_n])$  and hence it is well-defined. The claim is proved.  $\square$

Fix  $z \in \mathbb{R}^n$  such that the evaluation map  $B \mapsto \text{Det}(\rho(B))(z)$  belongs to  $L^1_{\text{loc}}$ . Next let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be defined by  $F(B) = c(B).B$ , where  $B \in \mathbb{R}^n$  and where  $c(B) \in \mathbb{R}$  is given by

$$c(B) = \int_0^B \text{Det}(\rho(x_1, \dots, x_n))(z) dx_1 \dots dx_n.$$

Here “0” stands for the origin of  $\mathbb{R}^n$  while the integration path is the radial line from the origin to  $B \in \mathbb{R}^n$ . The map  $F$  is clearly continuous and its restriction to real lines through the origin is monotone in the natural sense. It then follows that  $F$  is one-to-one and hence it is a homeomorphism onto its image contained in  $\mathbb{R}^n$ . Furthermore the homeomorphism  $F$  is, indeed, absolutely continuous since the map  $B \mapsto \text{Det}(\rho(B))(z)$  belongs to  $L^1_{\text{loc}}$ . Therefore the measure  $\nu = F^*\text{Leb}$  is absolutely continuous with respect to  $\text{Leb}$ . Finally,  $\nu$  is invariant by  $D^1[\rho_1((\mathbb{R}^*)^n \times \mathbb{R}^n)]$  since  $F$  conjugates  $D^1[\rho_1((\mathbb{R}^*)^n \times \mathbb{R}^n)]$  to the group of translations of  $\mathbb{R}^n$  and the latter preserves  $\text{Leb}$ . In other words, the measure  $\nu$  fulfils the conditions in the statement.  $\square$

We can now complete the proof of Proposition 4.7.

*Proof of Proposition 4.7.* Consider the measure  $\nu$  whose existence is ensured by Lemma 4.8. The push-forward  $H_*\nu$  of  $\nu$  by  $H$  is a sigma-finite measure on  $\mathbb{R}^n$  that happens to be invariant by  $\mathbf{T}$ , i.e. by all translations of  $\mathbb{R}^n$ . Due to the standard uniqueness property of the Haar measure, it follows that  $H_*\nu$  must be a constant multiple of  $\text{Leb}$ . Thus  $H^*\text{Leb}$  is again an absolute continuous measure since it coincides with a constant multiple of  $\nu$ . This means that  $H$  is an absolutely continuous *homeomorphism*. Since  $\mu = H_*\text{leb}$ , with  $H$  absolutely continuous, we conclude that  $\mu$  must be absolutely continuous with respect to  $\text{Leb}$ . The proposition is proved.  $\square$

Here is an immediate corollary of Proposition 4.7 that is worth stating in view of the toy-theorem mentioned in the Introduction.

**Corollary 4.9.** *Assume that  $G$  is a dense subgroup of  $\text{PSL}(2, \mathbb{C})$  projectively acting on  $S^2$ . Suppose that  $\mu$  is a probability measure on  $S^2$  that happens to be quasi-invariant by closure of  $G$  in  $\text{PSL}(2, \mathbb{C})$ . Then  $\mu$  is absolutely continuous.*

*Proof.* By assumption the closure of  $G$  is all of  $\text{PSL}(2, \mathbb{C})$ . In particular it contains a copy of the affine group  $\text{Aff}(\mathbb{C})$  of the complex line. According to Proposition 4.7, a measure quasi-invariant by  $\text{Aff}(\mathbb{C})$  must be absolutely continuous. The statement follows at once.  $\square$

For the sake of completeness, we also mention the proof of Linear Theorem stated in the Introduction.

*Proof of Linear Theorem.* It is an immediate combination of Proposition 4.9 with Theorem D to be proved in Section 5.  $\square$

**4.3. Proof of Theorem A.** In what concerns the existence of vector fields in the closure of groups, the main result obtained in this paper is Theorem 4.2. Once one vector field in the  $C^\infty$ -closure of  $G$  does exist, further generic conditions imposed on the group  $G$  allow us to construct additional vector fields possessing similar properties. The choice of the desired “generic” conditions is rather flexible and does not constitute the main issue in most applications. Here we shall proceed from a slightly more conceptual point of view directly inspired by [Be-2].

As before, we fix a local coordinate about a contracting fixed point of  $F$  so that  $F$  becomes the homothety  $(x_1, \dots, x_n) \mapsto (\lambda_1 x_1, \dots, \lambda_n x_n)$ . Consider the (pseudo-) Lie algebra  $\mathcal{G}$  of analytic vector fields about  $p$  that are in the  $C^\infty$ -closure of  $G$ . As it will be seen, this Lie algebra can be *filtered* by means of  $F$ -homogeneous components. Concerning filtered and/or graded Lie algebras, the reader is referred to [D] for the relevant definitions and results used below. To begin with, denote by  $m^1(F) > m^2(F) > \dots$  the collection of absolute values of all complex numbers having the form  $\lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n} / \lambda_l$ , with  $\alpha_i \in \mathbb{N}$  and  $l \in \{1, \dots, n\}$ . Fixed  $j \in \mathbb{N}^*$ , denote by  $\mathcal{V}^j$  the vector space spanned by all vector monomials whose  $F$ -multipliers are equal to  $m^j(F)$ . As already pointed out,  $\mathcal{V}^j$  has finite dimension as vector space for every  $j \in \mathbb{N}$ . Next set  $\mathcal{G}^j = \mathcal{G} \cap \mathcal{V}^j$ . Owing to Lemma 4.6, it follows that  $\mathcal{G} = \bigoplus_{j \in \mathbb{N}^*} \mathcal{G}^j$  where each  $\mathcal{G}^j$  is a finite dimensional vector space. Next we claim that  $[X^{j_1}, X^{j_2}]$  lies in  $\mathcal{G}^{j_1+j_2}$  provided that  $X^{j_1} \in \mathcal{G}^{j_1}$  and  $X^{j_2} \in \mathcal{G}^{j_2}$ . This is an immediate consequence of the general formula  $F^*[X, Y] = [F^*X, F^*Y]$  which, in our case, yields  $F^*[X^{j_1}, X^{j_2}] = m^{j_1}(F)m^{j_2}(F)[X^{j_1}, X^{j_2}]$ , i.e.  $m^{j_1}(F)m^{j_2}(F)$  is the  $F$ -multiplier associated to  $[X^{j_1}, X^{j_2}]$ . An elementary counting of possibilities then shows that the multiplier in question cannot appear before the position  $j_1 + j_2$  in the list  $m^1(F), m^2(F), \dots$  what, in turn, establishes the claim.

Summarizing what precedes, we have proved the following.

**Proposition 4.10.** *The Lie (pseudo) algebra of analytic vector fields defined about  $p$  and contained in the  $C^\infty$ -closure of  $G$  is graded.*  $\square$

So far, we have always worked with a subgroup  $G \subset \text{Diff}^\omega(M)$  satisfying conditions (1), (2), (3) of Section 2. Let us now assume that condition (4) is also satisfied. Then we have.

**Lemma 4.11.** *Suppose that  $G \subset \text{Diff}^\omega(M)$  satisfies conditions (1) - (4) of Section 2. Then, for every  $q \in M$ , there are vector fields  $X_1, \dots, X_n$  in the  $C^\infty$ -closure of  $G$  such that  $X_1(q), \dots, X_n(q)$  form a basis for  $T_q M$  (in particular these vector fields are defined about  $q$ ).*

*Proof.* In view of Lemma 4.1, it suffices to prove the lemma for a contracting fixed point  $p$  of  $F$ . Therefore let us suppose for a contradiction that the statement is false. Let  $X_1, \dots, X_k$  be a *maximal* set of analytic vector fields in the  $C^\infty$ -closure of  $G$  about  $p$  that happen to be linearly independent at some point near  $p$ . Proposition 4.2 ensures that  $k \geq 1$ . Furthermore the (possibly singular) plane distribution  $\mathcal{D}$  associated to  $X_1, \dots, X_k$  is integrable since vector fields in the closure of  $\mathcal{G}$  form a Lie algebra and  $k$  is maximal.

*Claim.* We have  $k = n$ .

*Proof of the Claim.* Suppose we had  $k < n$  (strictly). The maximality of  $k$  implies that  $\mathcal{D}$  must be fully invariant by  $G$  and hence it induces a singular foliation  $\mathcal{F}$  on the open set  $U \subset M$  corresponding to the  $G$ -orbit of some small neighborhood of  $p$ . Next note that  $\mathcal{F}$  is non-singular otherwise its singular set would be a proper analytic set of  $M$  invariant by  $G$  what is impossible in view of condition (3). Nonetheless the existence of a non-singular foliation as above contradicts condition (4). We then conclude that  $k$  cannot be smaller than  $n$  proving the statement.  $\square$

Now consider vector fields  $X_1, \dots, X_n$  that are contained in the  $C^\infty$ -closure of  $G$  on a neighborhood of  $p$  and that are linearly independent at generic points in this neighborhood. If the vector fields  $X_1, \dots, X_n$  can be chosen linearly independent at  $p$ , then the lemma is proved. Suppose that this is not the case. Therefore the non-empty set of points  $q$  in the  $G$ -invariant set  $U \subset M$  at which the rank of the subspace of  $T_q M$  spanned by vector fields in the Lie algebra in question is less than  $n$  constitutes a proper analytic subset of  $M$  invariant by  $G$ . Again this contradicts condition (3). The lemma is proved.  $\square$

**Lemma 4.12.** *With the preceding notations, suppose that  $G \subset \text{Diff}^\omega(M)$  satisfies conditions (1) - (5) of Section 2. Then the Lie algebra  $\mathcal{G}$  consisting of analytic vector field in the  $C^\infty$ -closure of  $G$  about a contracting fixed point  $p$  contains all analytic vector fields defined about  $p$ .*

*Proof.* Note that  $\mathcal{G}$  is a graded Lie algebra in view of Lemma 4.6. It is also transitive thanks to Lemma 4.11. If the dimension of  $\mathcal{G}$  can be shown to be infinite, then it will also follow that  $\mathcal{G}$  is irreducible. These Lie algebras were classified by E. Cartan (note that vector fields in  $\mathcal{G}$  are analytic). From his classification it results that  $\mathcal{G}$  has to consist of all analytic vector fields about  $p$  since the remaining possibilities have finite dimension. The lemma will then be proved.

Let us then suppose for a contradiction that the dimension of  $\mathcal{G}$  is finite. We also fix a small neighborhood  $U$  of  $p$  and denote by  $G_U$  the pseudogroup of local diffeomorphism defined on  $U$  by restrictions of elements in  $G$ . Consider now the abstract finite dimensional Lie algebra  $\mathcal{L}$  isomorphic to  $\mathcal{G}$ . Every element  $g$  in  $G_U$  acts linearly on  $\mathcal{G}$  by pull-backs and, therefore, can be identified to an element of the automorphism group  $\text{Aut}(\mathcal{L})$  of  $\mathcal{L}$ . Now, note that  $\text{Aut}(\mathcal{L})$  is itself a Lie group of finite dimension and the pseudogroup  $\Gamma_U$  associated to the image of  $G_U$  in  $\text{Aut}(\mathcal{L})$  can be considered. Similarly we denote by  $\Gamma$  the group arising from the global

realization of  $\Gamma_U$ . However, by construction, there is a local action of  $\Gamma_U$  on  $U$ . Furthermore, this local action can naturally be extended to the closure  $\overline{\Gamma}_U$  of  $\Gamma_U$  in  $\text{Aut}(\mathcal{L})$ . However  $\overline{\Gamma}_U$  is contained in the closure  $\overline{\Gamma}$  of  $\Gamma$  and  $\overline{\Gamma}$  is, in its own right, a finite dimensional Lie group since it is a closed subgroup of  $\text{Aut}(\mathcal{L})$ . Assembling all this information, we conclude that  $G$  is locally conjugate about  $p$  to a finite dimensional Lie group. This however contradicts condition (5) and establishes the lemma.  $\square$

*Proof of Theorem A.* Suppose that  $\mu$  is a probability measure on  $M$  quasi-invariant under the  $C^\infty$ -close of  $G$ . Standard arguments based on Lemma 4.11 makes it clear that the orbit under  $G$  of every point  $q \in M$  is dense in  $M$ . Similarly,  $G$  is ergodic with respect to the Lebesgue measure on  $M$ . So, from now on, we assume for a contradiction that  $\mu$  is singular with respect to the Lebesgue measure on  $M$ . Since  $G$  is ergodic, there is a Borel set  $\mathcal{B} \subset M$  having null Lebesgue measure and satisfying  $\mu(\mathcal{B}) = 1$ . Besides, if  $U$  is a non-empty open subset of  $M$ , then  $\mu(\mathcal{B} \cap U) > 0$  since  $M$  is covered by the  $G$ -orbit of  $U$ .

Consider now a contracting fixed point  $p$  for  $F$ . Let  $U$  be a small neighborhood of  $p$  identified to all of  $\mathbb{R}^n$  through some local  $C^\infty$ -coordinate  $\varphi$ . Modulo normalizing the restriction of  $\mu$  to  $U$ , we can assume that  $U$  is equipped with a probability measure  $\nu$  which is quasi-invariant by the pseudogroup  $\overline{G}_U$  induced by the restriction to  $U$  of all elements in the  $C^\infty$ -closure of  $G$ . By setting  $\mathcal{B}' = \mathcal{B} \cap U$ , we have  $\nu(\mathcal{B}') = 1$  whereas the Lebesgue measure of  $\mathcal{B}'$  equals zero.

By means of the coordinate  $\varphi$ ,  $\overline{G}_U$  becomes identified to a pseudogroup of  $C^\infty$ -diffeomorphisms of  $\mathbb{R}^n$  containing the contraction corresponding to  $F$  as well as the whole translation group  $\mathbf{T}$ , cf. Lemma 4.12. It follows from Proposition 4.7 that  $\nu$  must be absolute continuous since it is quasi-invariant by the latter pseudogroup. The resulting contradiction establishes the theorem.  $\square$

Now we have:

*Proof of Corollary B.* Let  $\mu$  be as in the statement and consider a Borel set  $\mathcal{B} \subset S^1$  such that  $\mu(\mathcal{B}) > 0$ . Since  $\mu$  is supposed to be a  $d$ -quasi-volume for  $G$ , there is a uniform constant  $C > 0$  such that

$$(12) \quad \mu(g(\mathcal{B})) \geq C \int_{\mathcal{B}} \|\text{Jac}[Dg](x)\|^d d\mu$$

for every  $g \in G$ . Next let  $\overline{g} : U \subseteq M \rightarrow M$  be a diffeomorphism in the closure of  $G$  and defined on an open set  $U$  containing  $\mathcal{B}$ . To conclude that  $\mu$  is quasi-invariant by elements in the  $C^\infty$ -closure of  $G$ , we need to show that  $\mu(\overline{g}(\mathcal{B})) > 0$ . For this, fix a sequence  $\{g_i\}$  of actual elements in  $G$  whose restrictions  $g_{i|U}$  to  $U$  converge in the  $C^\infty$ -topology to  $\overline{g}$  (actually  $C^1$ -convergence would be enough in the sequel). It follows, in particular, that  $\mu(\overline{g}(\mathcal{B})) = \lim_{i \rightarrow \infty} \mu(g_{i|U}(\mathcal{B}))$ . Equation (12) and the *uniform* convergence of  $\text{Jac}[Dg_{i|U}]$  towards  $\text{Jac}[D\overline{g}]$  yields the estimate

$$\mu(\overline{g}(\mathcal{B})) = \lim_{i \rightarrow \infty} \mu(g_{i|U}(\mathcal{B})) \geq C \lim_{i \rightarrow \infty} \int_{\mathcal{B}} \|\text{Jac}[Dg_{i|U}]\|^d d\mu = C \int_{\mathcal{B}} \|\text{Jac}[D\overline{g}]\|^d d\mu.$$

The last integral is clearly strictly positive since  $\overline{g}$  is a diffeomorphism onto its image. Hence  $\mu(\overline{g}(\mathcal{B})) > 0$  and thus  $\mu$  is quasi-invariant by elements in the  $C^\infty$ -closure of  $G$ . In view of Theorem A, we conclude that  $\mu$  is absolutely continuous.  $\square$

Similarly, let us now prove Corollary C.

*Proof of Corollary C.* The foliations  $\mathcal{F}$  on  $\mathbb{C}P(n+1)$  constructed in [L-R] are such that there exists a local transverse section  $\Sigma$  satisfying the following conditions (cf. Theorem 9.3 in [L-R]):

- (1) The section  $\Sigma$  is isomorphic to a poly-disc in  $\mathbb{C}^n$ .
- (2) Every (regular) leaf  $L$  of  $\mathcal{F}$  intersects  $\Sigma$
- (3) The pseudogroup  $G_\Sigma$  of local diffeomorphisms of  $\Sigma$  induced by the holonomy pseudogroup of  $\mathcal{F}$  on  $\Sigma$  has “large affine part”, i.e. in suitable coordinates its closure contains a copy of  $\mathrm{SL}(n, \mathbb{C}) \ltimes \mathbb{C}^n$ .

For the reasons already explained, every transverse  $d$ -quasi-volume for  $\mathcal{F}$  must induce a finite (non-zero) measure  $\mu$  on  $\Sigma$  that will be a  $d$ -quasi-volume for  $G_\Sigma$ . In particular,  $\mu$  must be quasi-invariant for the closure of  $G_\Sigma$  and hence, in suitable coordinates, by the natural action of  $\mathrm{SL}(n, \mathbb{C}) \ltimes \mathbb{C}^n$ . Proposition 4.7 then shows that  $\mu$  must be absolutely continuous. Corollary C is proved.  $\square$

## 5. THE RIGIDITY PHENOMENON

Let us begin the discussion of the rigidity phenomenon associated to groups as above by proving Theorem D. Let  $G_1, G_2$  be subgroups of  $\mathrm{Diff}^\omega(M)$  as in the statement of Theorem D. Denote by  $H$  a continuous orbit equivalence between the actions of  $G_1, G_2$ . According to the main result of [F-W], the homeomorphism  $H$  is equivariant and hence it constitutes a topological conjugacy between the actions in question. In fact, the only use made of condition (6) in Section 2 is to allow us to resort to the mentioned theorem of Fisher and Whyte.

Thus we have a topological conjugacy  $H$  between the actions of  $G_1, G_2$ , in other words, given  $g \in G_2$ , the homeomorphism  $H^{-1} \circ g \circ H$  actually coincides with an element of  $G_1$ . Theorem D can then be rephrased as follows.

**Theorem 5.1.** *Let  $G_1, G_2$  be as in the statement of Theorem D. Every homeomorphism  $H : M \rightarrow M$  conjugating the dynamics of  $G_1$  and  $G_2$  coincides with an element of  $\mathrm{Diff}^\omega(M)$ .*

Yet, Theorem 5.1 bears a fundamental difference from the analogous “topological rigidity theorems” proved, for example, in [R-1], [Be-2] and [R-3]. Namely, the present statement makes no assumption on  $H$  being “close to the identity” (though we assume that  $G_1$  acts “linearly” on  $M$ ). More precisely, if  $g_2 \in G_2$  is an element (say belonging to a fixed generating set for  $G_2$ ) that happens to be “close to the identity” (for a chosen topology), nothing ensures that  $g_1 \in G_1$  given by  $g_1 = H^{-1} \circ g_2 \circ H$  is “close to the identity” as well.

The key point for all these rigidity theorems consists of constructing vector fields *conjugated by  $H$*  (i.e. the vector fields are synchronized by  $H$ ). Recall that two (local) analytic vector fields  $X^1, X^2$  are said to be conjugated by  $H$  if the equation

$$(13) \quad H \circ \phi_{X^1}^t(x) = \phi_{X^2}^t(H(x))$$

is satisfied whenever both sides are defined, where  $\phi_{X^1}^t, \phi_{X^2}^t$  denote the local flows respectively associated to  $X^1, X^2$ . In fact, the fundamental result for the proof of Theorem D is the following proposition:

**Proposition 5.2.** *Fix a point  $q \in M$ . Then there are analytic vector fields  $X^1, X^2$  with  $X^1$  defined about  $q$  and  $X^2$  defined about  $H(q)$  such that the conditions below are satisfied.*

- (1)  $X^1$  (resp.  $X^2$ ) is contained in the  $C^1$ -closure of  $G_1$  (resp.  $G_2$ ). Besides  $X^1(q) \neq 0$  (resp.  $X^2(H(q)) \neq 0$ ).
- (2) If  $\phi_{X^1}^t$  (resp.  $\phi_{X^2}^t$ ) denotes the local flow associated to  $X^1$  (resp.  $X^2$ ), then Equation (13) holds whenever both sides are defined.

Assuming Proposition 5.2 holds, the proof of Theorem D can be derived as follows.



*Proof of Theorem D.* Fix  $q \in M$  and vector fields  $X^1, X^2$  as in Proposition 5.2. In view of the discussion conducted in Section 4.3, the group  $G_1$  acts on  $X^1$  to yield a collection of vector fields  $X^1 = X_1^1, \dots, X_n^1$  contained in the closure of  $G$  and such that the vectors  $X_1^1(q), \dots, X_n^1(q)$  form a basis for  $T_q M$ . However, to every element  $g_1 \in G_1$  acting on  $X^1$  there corresponds an element  $g_2 = H \circ g_1 \circ H^{-1} \in G_2$  acting on  $X^2$ . It then follows the existence of vector fields  $X^2 = X_1^2, \dots, X_n^2$  fulfilling the following conditions:

- $X_1^2, \dots, X_n^2$  are analytic vector fields (contained in the  $C^1$ -closure of  $G$ ).
- $X_1^2(H(q)), \dots, X_n^2(H(q))$  form a basis for  $T_{H(q)} M$ .
- For every  $i \in \{1, \dots, n\}$ ,  $H$  conjugates the vector fields  $X_i^1$  and  $X_i^2$ .

Consider now the local analytic coordinates  $(s_1, \dots, s_n)$  (resp.  $(\bar{s}_1, \dots, \bar{s}_n)$ ) about  $q$  (resp.  $H(q)$ ) defined by

$$(s_1, \dots, s_n) \mapsto \phi_{X_n^1}^{s_n} \circ \dots \circ \phi_{X_1^1}^{s_1}$$

(resp.  $(\bar{s}_1, \dots, \bar{s}_n) \mapsto \phi_{X_n^2}^{\bar{s}_n} \circ \dots \circ \phi_{X_1^2}^{\bar{s}_1}$ ). Since  $H$  conjugates the vector fields  $X_i^1$  and  $X_i^2$  for every  $i \in \{1, \dots, n\}$ , it becomes clear that  $H$ , mapping  $q$  to  $H(q)$ , locally coincides with the identity in the mentioned coordinates. Thus  $H$  is analytic about  $q$ . Since  $q$  is an arbitrary point of  $M$ , the statement of Theorem D follows at once.  $\square$

Let us now supply the proof of Proposition 5.2 by exploiting the “linear” character of the action of  $G_1$  on  $M$ .

*Proof of Proposition 5.2.* We shall mainly work with the “non-linear” group  $G_2$ . As always, it suffices to prove the statement on a neighborhood of a contracting fixed point  $p$  of some Morse-Smale element of  $G_2$ .

For this, note that the proof of the proposition in question requires us to construct two conjugate analytic vector fields: whether or not they belong to the closure of the corresponding groups  $G_1, G_2$  is no longer relevant. Thus we shall base our construction in Proposition 3.4 yielding analytic vector fields in the  $C^r$ -closure of  $G_2$  (actually we may work with  $r = 1$ ). Also it suffices to find  $X^1, X^2$  such that  $X^2$  is not identically zero (and hence neither is  $X^1$ ). The condition  $X^2(p) \neq 0$  can immediately be achieved by replacing the “original”  $X^2$  by  $g^* X^2$  for a suitable  $g \in G_2$  (recall that  $G_2$  has dense orbits in  $M$ ).

The construction of  $X^1, X^2$  goes as follows. Let  $\{h_i^2\}$  be a sequence of elements in  $G_2$  such that the vector fields  $X_i^2$  defined by

$$(14) \quad X_i^2(x) = \frac{1}{\sup_{x \in U} \|h_i^2(x) - x\|} \cdot (h_i^2(x) - x)$$

converge in the  $C^r$ -topology to a non-identically zero analytic vector field  $X^2$  in the  $C^r$ -closure of  $G_2$ .

Consider now the sequence  $\{h_i^1\}$  of elements in  $G_1$  given by  $h_i^1 = H^{-1} \circ h_i^2 \circ H$ . Clearly the sequence  $\{h_i^1\}$  converges *uniformly* to the identity on  $M$ . However, since  $G_1$  is “linear”, this sequence also converges to the identity in, say, the  $C^\infty$ -topology (it is an easy consequence of Lemma 3.3). Besides, this sequence also satisfies the assumptions of Proposition 3.1, cf. again Lemma 3.3). Thus, letting

$$(15) \quad X_i^1(x) = \frac{1}{\sup_{x \in U} \|h_i^1(x) - x\|} \cdot (h_i^1(x) - x).$$

Modulo passing to a subsequence, the vector fields  $X_i^1$  will converge in the  $C^r$ -topology towards an analytic vector field  $X^1$  in the  $C^r$ -closure of  $G_1$ . Since, for every  $i \in \mathbb{N}$ ,  $h_i^1 = H^{-1} \circ h_i^2 \circ H$ , it follows that  $H$  conjugates  $X^1$  to  $X^2$ . The proposition is proved.  $\square$

**5.1. Additional issues and questions on OE.** As promised, this paper will end with a few questions concerning the “rigidity phenomenon” and orbit equivalences. Let us start with a general question concerning the role of the “linear” assumption on  $G_1$  imposed by the statement of Theorem D. Note that this assumption was only used in the proof of Proposition 5.2. In most situations, the sequence  $\{g_j^2\}_{j \in \mathbb{N}}$  of elements in  $G^2$  converging to the identity and satisfying the conditions of Lemma 3.1 can be obtained by re-normalizing (by means of  $F$ ) a sequence of commutators. In these cases, an affirmative answer to the question below would allow us to remove the “linear” assumption from the corresponding statement (for details see the standard construction for example in [R-1] or [R-3]).

*Question: does  $C^0$ -convergence imply  $C^\infty$ -convergence.* To accurately state this question, let  $f, g$  be two diffeomorphisms of  $M$ . By taking  $\{f, g, f^{-1}, g^{-1}\}$  as the generating set  $S$  of a group  $G$ , consider the sequence of subsets  $S(j)$  defined by Ghys, cf. Section 2. We may assume that  $f, g$  are close to the identity so that every non-trivial sequence of diffeomorphisms  $\{g_k\}_{k \in \mathbb{N}}$  such that, say,  $G_k \in S(k)$ , converges to the identity (say in the  $C^\infty$ -topology).

Now let  $H$  denote a homeomorphism of  $M$  such that  $\bar{f} = H^{-1} \circ f \circ H$  and  $\bar{g} = H^{-1} \circ g \circ H$  are smooth/analytic diffeomorphism of  $M$ . Let  $\bar{S}(j)$  be the analogous sequence of commutators that is obtained by starting with  $\bar{S} = \{\bar{f}, \bar{g}, \bar{f}^{-1}, \bar{g}^{-1}\}$ . In particular, for every  $k \in \mathbb{N}$  and  $g_k \in S(k)$ , there corresponds a diffeomorphism  $\bar{g}_k \in \bar{S}(k)$ . Clearly the sequence of diffeomorphisms  $\{\bar{g}_k\}$  converges in the  $C^0$ -topology to the identity. In view of what precedes, it is natural to ask whether or not  $\{\bar{g}_k\}$  converges to the identity in the  $C^\infty$ -topology as well? This question appears to be interesting already when the ambient manifold  $M$  is the circle. Note, however, that the problem is immaterial in the holomorphic setting (even for pseudogroups) due to the Cauchy formula.

*Rigidity for measurable conjugacy.* Let  $G_1, G_2 \subset \text{Diff}^\omega(M)$  be as in the statement of Theorem D. Assume that  $\theta : M \rightarrow M$  is a measurable bijection conjugating (Lebesgue almost everywhere) the actions of  $G_1, G_2$ . Our purpose here is to sketch a proof that, possibly modulo some minor additional generic assumptions,  $\theta$  must coincide almost everywhere with an element of  $\text{Diff}^\omega(M)$ . The argument is borrowed from [R-2]. First consider the diagonal action  $G_1 \times G_2$  on  $M \times M$  and note that vector fields in the  $C^\infty$ -closure of this action can be constructed. In fact, it suffices to consider  $X^2$  as in the proof of Proposition 5.2 and obtain a vector field  $X^1$  by means of the sequence  $h_i^1 = \theta^{-1} \circ h_i^2 \circ \theta$ . Here the classical Lusin theorem applied to  $\theta$  suffices to conclude that  $\{h_i^1\}$  converges  $C^0$ , and hence  $C^\infty$ , to the identity. The resulting vector field  $X = (X^1, X^2)$  is then in the  $C^\infty$ -closure of  $G_1 \times G_2$ . Now note that the graph  $\text{Graph}(\theta)$  of  $\theta$  in  $M \times M$  is invariant under this diagonal action. The same argument given in Section 4 of [R-2] shows the following: a generic (Lebesgue density) point  $p_1$  in the domain of  $\theta$  is such that no vector field  $X = (X^1, X^2)$  in the  $C^\infty$ -closure of the diagonal action and defined about  $(p_1, \theta(p_1))$  is neither *vertical* nor *horizontal* at  $(p_1, \theta(p_1))$ . By definition this means that every vector field  $X = (X^1, X^2)$  in the  $C^\infty$ -closure of  $G_1 \times G_2$  and defined about  $(p_1, \theta(p_1))$  is such that either  $X^1(p_1), X^2(\theta(p_1))$  vanish simultaneously or they are both different from zero. The third step of the proof, for which generic assumptions may be required, consists of showing that the set of points  $(p_1, p_2) \in M \times M$  such that no vector field defined about  $(p_1, p_2)$  and contained in the  $C^\infty$ -closure of  $G_1 \times G_2$  is either vertical or horizontal at  $(p_1, p_2)$  constitutes an analytic subset of dimension  $n$  of  $M \times M$ . Once this last statement is proved, it follows that

Graph( $\theta$ ) must be contained in the analytic set in question so that the regular character of  $\theta$  can easily be derived.

*OE and further questions.* Our main interest in the above mentioned rigidity result lies in the fact that *it is easy to construct actions that are not  $C^\infty$ -conjugate* and hence *it becomes easy* to construct actions that are not measurably conjugate. A general question is then as follows: can we use this result to construct new examples of non-OE actions for a same abstract group?

There are many difficult questions concerning OE between actions (see for example [Fu]) despite some rather spectacular progress made in the past decade or so. Yet, constructing examples of non-OE actions for a same group with as much regularity as possible is still an interesting question. If the “very regular” actions for which the “rigidity of measurable conjugacies” holds are considered, we may wonder when it is possible to ensure that a given OE must necessarily be equivariant. Already in the case of subgroups of  $\mathrm{PSL}(2, \mathbb{C})$  projectively acting on the sphere  $S^2$ , some specific question can explicitly be stated, namely:

- According to Sullivan [Su], every Kleinian group is OE to a  $\mathbb{Z}$ -action and hence the action of every two Kleinian groups are OE (in the context of quasi-invariant measures). The proof depends heavily on the fact that these groups are discrete. So we may wonder how many non-OE actions can be produced by means of, say the free group on two generators, realized as dense subgroups of  $\mathrm{PSL}(2, \mathbb{C})$ .
- For which classes of groups, if any, admitting non-discrete actions every OE must be equivariant?
- Related to the last question above, is there a generalization of the main result in [F-W] to the measure-theoretic context? Whereas this extension seems unlikely to exist, almost every non-trivial generalization of the criterion in [F-W] would be very interesting.

## REFERENCES

- [Ar-II] V. ARNOLD & Y. IL'YASHENKO, *Ordinary differential equations*, EMS-Dynamical Systems I, (1988), Springer-Verlag, 1-148.
- [Bt] R. BARTLE, *The Elements of Integration and Lebesgue measure*, Wiley Classics Library, (1995)
- [Be-1] M. BELLART, Sur certains pseudogroupes de biholomorphismes locaux de  $(\mathbb{C}^n, 0)$ , *Bull. Soc. Math. France* **129**, (2001), 259-284.
- [Be-2] M. BELLART, On the dynamics of certain actions of free groups on closed real analytic manifolds, *Comment. Math. Helv.* **77**, (2002), 524-548.
- [B-T] H. BROER, & F. TANGHERMAN, From a differentiable to a real analytic perturbation theory, applications to the Kupka Smale theorems, *Ergodic Theory Dynam. Systems*, **6**, (1986), 345-362.
- [C-S-V] I. CORNFELD, Y. SINAI & A. VERSHIK, *General Theory of Groups of Measure Preserving Transformations*, EMS-Dynamical Systems II, (1989), Springer-Verlag, 1-98.
- [D] M. DEMAURE, Classification des algèbres de Lie filtrées, *Sém. Bourbaki* **326** (1967).
- [F-W] D. FISHER & K. WHYTE, When is a group action determined by its orbit structure?, *Geom. funct. anal.*, **13**, (2003), 1189-1200.
- [Fu] A. FURMAN, A survey on measured group theory, *Geometry, rigidity and group actions*, Chicago Lectures in Math., Univ. Chicago Press, Chicago IL, (2011), 296-374.
- [G] E. GHYS, Sur les Groupes Engendrés par des Difféomorphismes Proches de l'Identité, *Bol. Soc. Bras. Mat.* **24**, 2, (1993), 137-178.
- [G-S] E. GHYS & V. SERGIESCU, Sur un groupe remarquable de difféomorphismes du cercle, *Comment. Math. Helv.*, **62**, (1987), 185-239.
- [G-P-S] T. GIORDANO, I. PUTNAM & C. SKAU, Topological orbit equivalence and  $C^*$ -crossed products, *J. reine angew. Math.*, **469**, (1995), 51-111.
- [Il-1] Y. IL'YASHENKO, The topology of phase portraits of analytic differential equations in the complex projective plane (Russian) *Trudy Sem. Petrovsk.*, **4**, (1978), 83-136, Translated in *Sel. Math. Sov.*, **5**, 2, (1986), 141-199.

- [K-LP] V. KAIMANOVICH & V. LE PRINCE, Random matrix products with singular harmonic measure, *Geom. Dedicata*, **150**, (2011), 257-279.
- [K-L] V. KAIMANOVICH & M. LYUBICH, Conformal and harmonic measures on laminations associated to rational maps, *Memoirs of the AMS*, **173**, 820, (2005).
- [L-R] F. LORAY & J.C. REBELO, Minimal, rigid foliations by curves on  $\mathbb{CP}(n)$ , *J. Eur. Math. Soc.*, **5**, 2, (2003), 147-201.
- [L-M] M. LYUBICH & Y. MINSKY, Laminations in holomorphic dynamics, *J. Diff. Geom.*, **47**, (1997), 17-94.
- [Na] NAKAI, I. Separatrix for Non Solvable Dynamics on  $\mathbb{C}, 0$ , *Ann. Inst. Fourier*, **44**, 2, (1994), 569-599.
- [O-U] J. OXTOBI & S. ULAM, Measure-preserving homeomorphisms and metrical transitivity, *Annals of Math.*, **42**, 4, (1941), 874-920.
- [R-1] J.C. REBELO, Ergodicity and rigidity for certain subgroups of  $\text{Diff}^\omega(\mathbb{R}, 0)$ , *Ann. Sci. École Norm. Sup. (4)*, **32**, 4, (1999), 433-453.
- [R-2] J.C. REBELO, A theorem of measurable rigidity in  $\text{Diff}^\omega(\mathbb{R}, 0)$ , *Ergodic Theory Dynam. Systems*, **21**, 5, (2001), 1525-1561.
- [R-3] J.C. REBELO, On the dynamics of generic non-Abelian free actions, *Bull. Braz. Math. Soc.*, **35**, 2, (2004), 211-251.
- [R-4] J.C. REBELO, Subgroups of  $\text{Diff}_+^\infty(S^1)$  acting transitively on unordered 4-tuples, *Transactions of the American Mathematical Society*, **356**, 11, (2004), 4543-4557.
- [R-5] J.C. REBELO, On the structure of quasi-invariant measures for non-discrete subgroups of  $\text{Diff}^\omega(S^1)$ , *to appear in Proc. London Math. Soc.*, 33 pages.
- [Sh-1] SHCHERBAKOV, A.A., On the density of an orbit of a pseudogroup of conformal mappings and a generalization of the Hudai-Verenov theorem, *Vestnik Morskovskogo Universiteta Matematika*, **31**, 4, (1982), 10-15.
- [Sh-2] SHCHERBAKOV, A.A., Topological and analytic conjugation of non commutative groups of conformal mappings *Trudy Sem. Petrovsk*, **10**, (1984), 170-192.
- [Su] D. SULLIVAN, Discrete Conformal Groups and Measurable Dynamics, *Bulletin of the AMS (New Series)*, **6**, 1, (1982), 57- 73.

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